5 The Integral

5.1 Area
5.2 The Definite Integral
5.3 The Fundamental Theorem of Calculus
5.4 Properties of the Definite Integral
5.5 The Indefinite Integral; Growth and Decay Models
5.6 Method of Substitution; Newton's Law of Cooling
Chapter Review
Chapter Project

Managing the Klamath River

The Klamath River starts in the eastern lava plateaus of Oregon, passes through a farming region of that state, and then crosses northern California. Because it is the largest river in the region, and runs through a variety of terrain and has different land uses, it is important in a number of ways. Historically, the Klamath supported large salmon runs. It is used to irrigate agricultural lands, and to generate electric power for the region. Downstream, it runs through federal wild lands, and is used for fishing, rafting, and kayaking.

If a river is to be well-managed for such a variety of uses, its flow must be understood. For that reason, the U.S. Geological Survey (USGS) maintains a number of gauges along the Klamath. These typically measure the depth of the river, which can then be expressed as a rate of flow in cubic feet per second (ft^3/s). In order to understand the river flow fully, one must be able to find the total amount of water that flows down the river over any period of time.

CHAPTER 5 PROJECT The Chapter Project on page 61 examines ways to obtain the total flow over any period of time from data that provide flow rates.

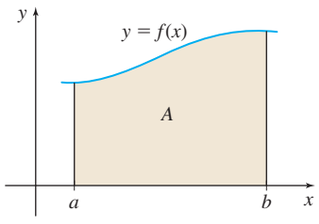


Figure 1 A is the area enclosed by the graph of f, the x-axis, and the lines $x = a$ and $x = b$.

We began our study of calculus by asking two questions from geometry. The first, the tangent problem, "What is the slope of the tangent line to the graph of a function?" led to the derivative of a function.

The second question was the area problem: Given a function f, defined and nonnegative on a closed interval $[a, b]$, what is the area enclosed by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$? Figure 1 illustrates this area.

The first two sections of Chapter 5 show how the concept of the integral evolves from the area problem. At first glance, the area problem and the tangent problem look quite dissimilar. However, much of calculus is built on a surprising relationship between the two problems and their associated concepts. This relationship is the basis for the *Fundamental Theorem of Calculus,* discussed in Section 5.3.

5.1 Area

OBJECTIVES *When you finish this section, you should be able to:*

1 Approximate the area under the graph of a function (p. 2)

2 Find the area under the graph of a function (p. 6)

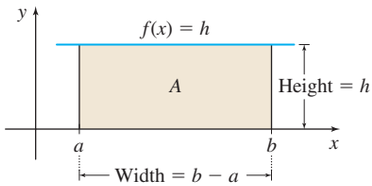


Figure 2 $A = h(b - a)$.

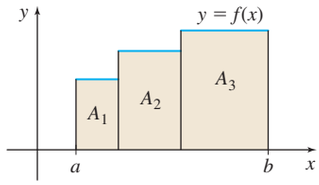


Figure 3 $A = A_1 + A_2 + A_3$

In this section, we present a method for finding the area enclosed by the graph of a function $y = f(x)$ that is nonnegative on a closed interval $[a, b]$, the x-axis, and the lines $x = a$ and $x = b$. The presentation uses summation notation ($\sum$), which is reviewed in Appendix A.5.

The area A of a rectangle with width w and height l is given by the geometry formula

$$\boxed{A = lw}$$

See Figure 2. The graph of a constant function $f(x) = h$, for some positive constant h, is a horizontal line that lies above the x-axis. The area enclosed by this line, the x-axis, and the lines $x = a$ and $x = b$ is the rectangle whose area A is the product of the width $(b - a)$ and the height h.

$$\boxed{A = h(b - a)}$$

If the graph of $y = f(x)$ consists of three horizontal lines, each of positive height as shown in Figure 3, the area A enclosed by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is the sum of the rectangular areas A_1, A_2, and A_3.

1 Approximate the Area Under the Graph of a Function

EXAMPLE 1 **Approximating the Area Under the Graph of a Function**

Approximate the area A enclosed by the graph of $f(x) = \frac{1}{2}x + 3$, the x-axis, and the lines $x = 2$ and $x = 4$.

Solution Figure 4 illustrates the area A to be approximated.

We begin by drawing a rectangle of width $4 - 2 = 2$ and height $f(2) = 4$. The area of the rectangle, $2 \cdot 4 = 8$, approximates the area A, but it underestimates A, as seen in Figure 5(a).

Alternatively, A can be approximated by a rectangle of width $4 - 2 = 2$ and height $f(4) = 5$. See Figure 5(b). This approximation of the area equals $2 \cdot 5 = 10$, but it overestimates A. We conclude that

$$8 < A < 10$$

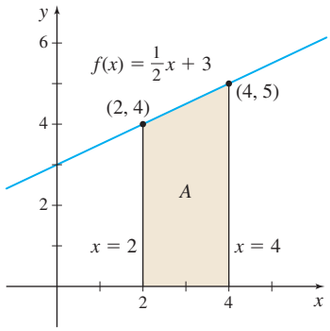
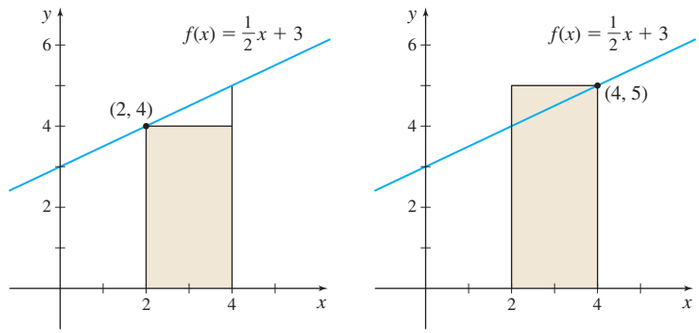


Figure 4



(a) The area A is underestimated. (b) The area A is overestimated.

Figure 5

The approximation of the area A can be improved by dividing the closed interval $[2, 4]$ into two subintervals, $[2, 3]$ and $[3, 4]$. Now we draw two rectangles: one rectangle with width $3 - 2 = 1$ and height $f(2) = \frac{1}{2} \cdot 2 + 3 = 4$; the other rectangle with width $4 - 3 = 1$ and height $f(3) = \frac{1}{2} \cdot 3 + 3 = \frac{9}{2}$. As Figure 6(a) illustrates, the sum of the areas of the two rectangles

$$1 \cdot 4 + 1 \cdot \frac{9}{2} = \frac{17}{2} = 8.5$$

underestimates the area.

Now we repeat this process by drawing two rectangles, one of width 1 and height $f(3) = \frac{9}{2}$; the other of width 1 and height $f(4) = \frac{1}{2} \cdot 4 + 3 = 5$. As Figure 6(b) illustrates, the sum of the areas of these two rectangles,

$$1 \cdot \frac{9}{2} + 1 \cdot 5 = \frac{19}{2} = 9.5$$

overestimates the area. We conclude that

$$8.5 < A < 9.5$$

obtaining a better approximation to the area.

NOTE The actual area in Figure 4 is 9 square units, obtained by using the formula for the area A of a trapezoid with base b and parallel heights h_1 and h_2:

$$A = \frac{1}{2}b(h_1 + h_2) = \frac{1}{2}(2)(4 + 5) = 9.$$

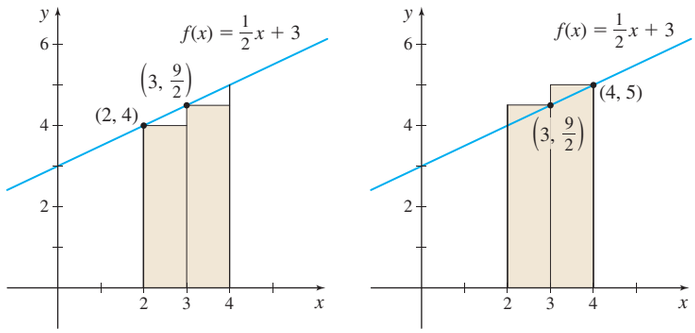


Figure 6

Observe that as the number n of subintervals of the interval increases, the approximation of the area A improves. For $n = 1$, the error in approximating A is 1 square unit, but for $n = 2$, the error is only 0.5 square unit.

NOW WORK **Problem 5.**

In general, the procedure for approximating the area A is based on the idea of summing the areas of rectangles. We shall refer to the area A enclosed by the graph of a function $y = f(x) \geq 0$, the x-axis, and the lines $x = a$ and $x = b$ as the **area under the graph of f from a to b**.

We make two assumptions about the function f:

- f is continuous on the closed interval $[a, b]$.
- f is nonnegative on the closed interval $[a, b]$.

We divide, or **partition**, the interval $[a, b]$ into n nonoverlapping subintervals:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$$

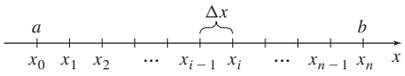


Figure 7

each of the same length. See Figure 7. Since there are n subintervals and the length of the interval $[a, b]$ is $b - a$, the common length Δx of each subinterval is

$$\Delta x = \frac{b-a}{n}$$

NEED TO REVIEW? The Extreme Value Theorem is discussed in Section 4.2, pp. xx–xx.

Since f is continuous on the closed interval $[a, b]$, it is continuous on every subinterval $[x_{i-1}, x_i]$ of $[a, b]$. By the Extreme Value Theorem, there is a number in each subinterval where f attains its absolute minimum. Label these numbers c_1, c_2, $c_3, \ldots, c_n$, so that $f(c_i)$ is the absolute minimum value of f in the subinterval $[x_{i-1}, x_i]$. Now construct n rectangles, each having the Δx as its base and $f(c_i)$ as its height, as illustrated in Figure 8. This produces n narrow rectangles of uniform width $\Delta x = \dfrac{b-a}{n}$ and heights $f(c_1), f(c_2), \ldots, f(c_n)$, respectively. The areas of the n rectangles are

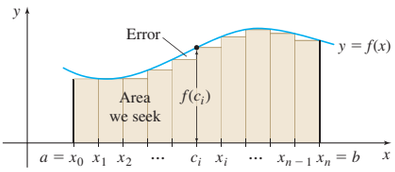


Figure 8 $f(c_i)$ is the absolute minimum value of f on $[x_{i-1}, x_i]$.

$$\text{Area of the first rectangle} = f(c_1)\Delta x$$
$$\text{Area of the second rectangle} = f(c_2)\Delta x$$
$$\vdots$$
$$\text{Area of the } n\text{th (and last) rectangle} = f(c_n)\Delta x$$

The sum s_n of the areas of the n rectangles approximates the area A. That is,

$$A \approx s_n = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_i)\Delta x + \cdots + f(c_n)\Delta x = \sum_{i=1}^{n} f(c_i)\Delta x$$

NEED TO REVIEW? Summation notation is discussed in Appendix A.5, pp. A-38 to A-43.

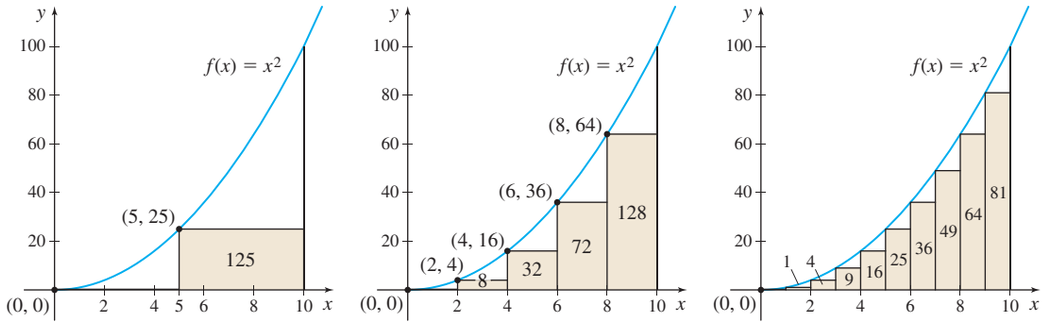
Since the rectangles used to approximate the area A lie under the graph of f, the sum s_n, called a **lower sum**, *underestimates* A. That is, $s_n \le A$.

EXAMPLE 2 Approximating Area Using Lower Sums

Approximate the area A under the graph of $f(x) = x^2$ from 0 to 10 by using lower sums s_n (rectangles that lie under the graph) for:

(a) $n = 2$ subintervals **(b)** $n = 5$ subintervals **(c)** $n = 10$ subintervals

Solution **(a)** For $n = 2$, we partition the closed interval $[0, 10]$ into two subintervals $[0, 5]$ and $[5, 10]$, each of length $\Delta x = \dfrac{10-0}{2} = 5$. See Figure 9(a). To compute s_2, we need to know where f attains its minimum value in each subinterval. Since f is an



(a) Two subintervals
$s_2 = 125$ sq. units

(b) Five subintervals
$s_5 = 240$ sq. units

(c) Ten subintervals
$s_{10} = 285$ sq. units

Figure 9

increasing function, the absolute minimum is attained at the left endpoint of each subinterval. So, for $n = 2$, the minimum of f on $[0, 5]$ occurs at 0 and the minimum of f on $[5, 10]$ occurs at 5. The lower sum s_2 is

$$s_2 = \sum_{i=1}^{2} f(c_i)\Delta x = \Delta x \sum_{i=1}^{2} f(c_i) = 5[f(0) + f(5)] = 5(0 + 25) = 125$$

$\Delta x = 5$; $f(c_1) = f(0)$; $f(c_2) = f(5)$; $f(0) = 0$; $f(5) = 25$

(b) For $n = 5$, partition the interval $[0, 10]$ into five subintervals $[0, 2]$, $[2, 4]$, $[4, 6]$, $[6, 8]$, $[8, 10]$, each of length $\Delta x = \dfrac{10 - 0}{5} = 2$. See Figure 9(b). The lower sum s_5 is

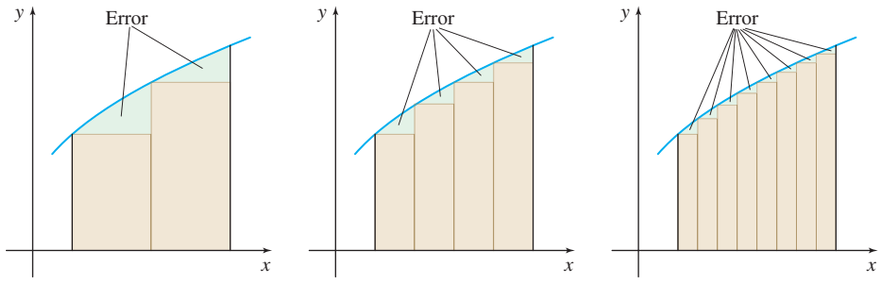
$$s_5 = \sum_{i=1}^{5} f(c_i)\Delta x = \Delta x \sum_{i=1}^{5} f(c_i) = 2[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$= 2(0 + 4 + 16 + 36 + 64) = 240$$

(c) For $n = 10$, partition $[0, 10]$ into 10 subintervals, each of length $\Delta x = \dfrac{10 - 0}{10} = 1$. See Figure 9(c). The lower sum s_{10} is

$$s_{10} = \sum_{i=1}^{10} f(c_i)\Delta x = \Delta x \sum_{i=1}^{10} f(c_i) = 1[f(0) + f(1) + f(2) + \cdots + f(9)]$$
$$= 0 + 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 = 285$$ ■

NOW WORK Problem 13(a).

In general, as Figure 10(a) illustrates, the error due to using lower sums s_n (rectangles that lie below the graph of f) occurs because a portion of the area lies outside the rectangles. To improve the approximation of the area, we increase the number of subintervals. For example, in Figure 10(b), there are four subintervals and the error is reduced; in Figure 10(c), there are eight subintervals and the error is further reduced. So, by taking a finer and finer partition of the interval $[a, b]$, that is, by increasing n, the number of subintervals, without bound, we can make the sum of the areas of the rectangles as close as we please to the actual area. (A proof of this statement is usually found in books on advanced calculus.)



(a) Two subintervals (b) Four subintervals (c) Eight subintervals

Figure 10

In Section 5.2, we see that, for functions that are continuous on a closed interval, $\lim_{n\to\infty} s_n$ always exists. With this discussion in mind, we now define the area under the graph of a function f from a to b.

DEFINITION Area A Under the Graph of a Function from a to b

Suppose a function f is nonnegative and continuous on a closed interval $[a, b]$. Partition $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$, each of length

$$\Delta x = \frac{b-a}{n}$$

In each subinterval $[x_{i-1}, x_i]$, let $f(c_i)$ equal the absolute minimum value of f on this subinterval. Form the lower sums

$$s_n = \sum_{i=1}^{n} f(c_i)\Delta x = f(c_1)\Delta x + \cdots + f(c_n)\Delta x$$

The **area A under the graph of f from a to b** is the number

$$\boxed{A = \lim_{n\to\infty} s_n}$$

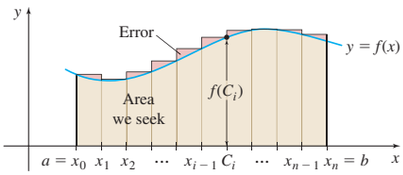


Figure 11 $f(C_i)$ is the absolute maximum value of f on $[x_{i-1}, x_i]$.

The area A is defined using lower sums s_n (rectangles that lie below the graph of f). By a parallel argument, we can choose values $C_1, C_2, \ldots, C_n$ so that the height $f(C_i)$ of the ith rectangle is the absolute maximum value of f on the ith subinterval, as shown in Figure 11. The corresponding **upper sums S_n** (rectangles that lie above the graph of f) *overestimate* the area A. So, $S_n \geq A$. It can be shown that as n increases without bound, the limit of the upper sums S_n equals the limit of the lower sums s_n. That is,

$$\boxed{\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = A}$$

2 Find the Area Under the Graph of a Function

In the next example, instead of using a specific number of rectangles to *approximate* area, we partition the interval $[a, b]$ into n subintervals, obtaining n rectangles. By letting $n \to \infty$, we find the *actual* area under the graph of f from a to b.

EXAMPLE 3 Finding Area Using Upper Sums

Find the area A under the graph of $f(x) = 3x$ from 0 to 10 using upper sums S_n (rectangles that lie above the graph of f). Then $A = \lim_{n\to\infty} S_n$.

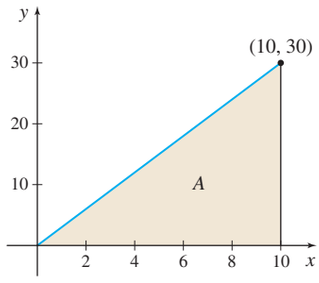


Figure 12 $f(x) = 3x, 0 \leq x \leq 10$

Solution Figure 12 illustrates the area A. We partition the closed interval [0, 10] into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, \quad [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_{n-1} < x_n = 10$$

and each subinterval is of length

$$\Delta x = \frac{10-0}{n} = \frac{10}{n}$$

The coordinates of the endpoints of each subinterval, written in terms of n, are

$$x_0 = 0, \; x_1 = \frac{10}{n}, \; x_2 = 2\left(\frac{10}{n}\right), \ldots, x_{i-1} = (i-1)\left(\frac{10}{n}\right),$$

$$x_i = i\left(\frac{10}{n}\right), \ldots, x_n = n\left(\frac{10}{n}\right) = 10$$

as illustrated in Figure 13.

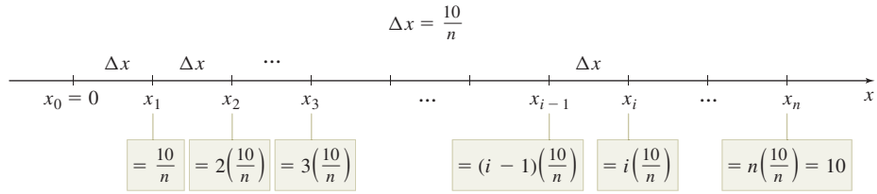


Figure 13

To find A using upper sums S_n (rectangles that lie above the graph of f), we need the absolute maximum value of f in each subinterval. Since $f(x) = 3x$ is an increasing function, the absolute maximum occurs at the right endpoint $x_i = i\left(\dfrac{10}{n}\right)$ of each subinterval. So,

$$S_n = \sum_{i=1}^{n} f(C_i)\Delta x = \sum_{i=1}^{n} 3x_i \cdot \frac{10}{n} = \sum_{i=1}^{n} \left(3 \cdot \frac{10i}{n}\right)\left(\frac{10}{n}\right) = \sum_{i=1}^{n} \frac{300}{n^2} i$$

$\Delta x = \dfrac{10}{n}$ $\quad$ $x_i = \dfrac{10i}{n}$

Using summation properties, we get

$$S_n = \sum_{i=1}^{n} \frac{300}{n^2} i = \frac{300}{n^2} \sum_{i=1}^{n} i = \frac{300}{n^2} \frac{n(n+1)}{2} = 150\left(\frac{n+1}{n}\right) = 150\left(1 + \frac{1}{n}\right)$$

Then

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[150\left(1 + \frac{1}{n}\right)\right] = 150 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 150$$

The area A under the graph of $f(x) = 3x$ from 0 to 10 is 150 square units. ■

NOTE The area found in Example 3 is that of a triangle. So, we can verify that $A = 150$ by using the formula for the area A of a triangle with base b and height h:

$$A = \frac{1}{2}bh = \frac{1}{2}(10)(30) = 150$$

NEED TO REVIEW? Summation properties are discussed in Appendix A.5, pp. A-38 to A-43.

RECALL $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

EXAMPLE 4 Finding Area Using Lower Sums

Find the area A under the graph of $f(x) = 16 - x^2$ from 0 to 4 by using lower sums s_n (rectangles that lie below the graph of f). Then $A = \lim\limits_{n\to\infty} s_n$.

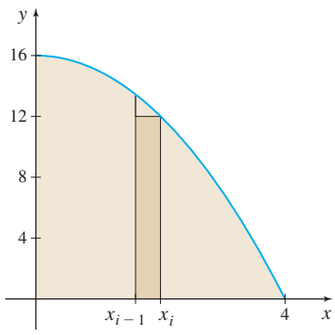


Figure 14 $f(x) = 16 - x^2,\ 0 \le x \le 4$

Solution Figure 14 shows the area under the graph of f and a typical rectangle that lies below the graph. We partition the closed interval $[0, 4]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < \cdots < x_i < \cdots < x_{n-1} < x_n = 4$$

and each interval is of length

$$\Delta x = \frac{4-0}{n} = \frac{4}{n}$$

As Figure 15 illustrates, the endpoints of each subinterval, written in terms of n, are

$$x_0 = 0,\ x_1 = 1\left(\frac{4}{n}\right),\ x_2 = 2\left(\frac{4}{n}\right), \ldots, x_{i-1} = (i-1)\left(\frac{4}{n}\right),$$

$$x_i = i\left(\frac{4}{n}\right), \ldots, x_n = n\left(\frac{4}{n}\right) = 4$$

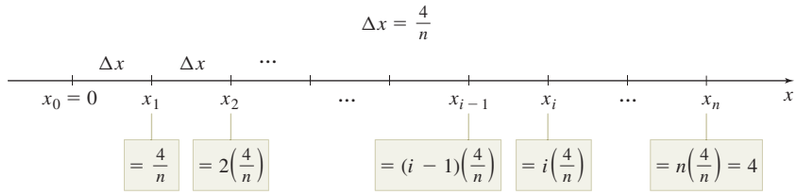


Figure 15

To find A using lower sums s_n (rectangles that lie below the graph of f), we must find the absolute minimum value of f on each subinterval. Since the function f is a decreasing function, the absolute minimum occurs at the right endpoint of each subinterval. So,

$$s_n = \sum_{i=1}^{n} f(c_i)\Delta x$$

Since $c_i = i\left(\frac{4}{n}\right) = \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$, we have

$$s_n = \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n}\left[16 - \left(\frac{4i}{n}\right)^2\right]\left(\frac{4}{n}\right) \qquad c_i = \frac{4i}{n};\ f(c_i) = 16 - c_i^2$$

$$= \sum_{i=1}^{n}\left[\frac{64}{n} - \frac{64i^2}{n^3}\right]$$

$$= \frac{64}{n}\sum_{i=1}^{n} 1 - \frac{64}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{64}{n}(n) - \frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = 64 - \frac{32}{3n^2}\left[2n^2 + 3n + 1\right]$$

$$= 64 - \frac{64}{3} - \frac{32}{n} - \frac{32}{3n^2} = \frac{128}{3} - \frac{32}{n} - \frac{32}{3n^2}$$

RECALL $\sum_{i=1}^{n} 1 = n$;
$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Then

$$A = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{128}{3} - \frac{32}{n} - \frac{32}{3n^2}\right) = \frac{128}{3}$$

The area A under the graph of $f(x) = 16 - x^2$ from 0 to 4 is $\frac{128}{3}$ square units. ■

NOW WORK **Problem 29.**

The previous two examples illustrate just how complex it can be to find areas using lower sums and/or upper sums. In the next section, we define an *integral* and show how it can be used to find area. Then in Section 5.3, we present the Fundamental Theorem of Calculus, which provides a relatively simple way to find area.

5.1 Assess Your Understanding

Concepts and Vocabulary

1. Explain how rectangles can be used to approximate the area enclosed by the graph of a function $y = f(x) \geq 0$, the x-axis, and the lines $x = a$ and $x = b$.
2. *True or False* When a closed interval $[a, b]$ is partitioned into n subintervals each of the same length, the length of each subinterval is $\frac{a+b}{n}$.
3. If the closed interval $[-2, 4]$ is partitioned into 12 subintervals, each of the same length, then the length of each subinterval is _____.
4. *True or False* If the area A under the graph of a function f that is continuous and nonnegative on a closed interval $[a, b]$ is approximated using upper sums S_n, then $S_n \geq A$ and $A = \lim_{n\to\infty} S_n$.

Skill Building

5. Approximate the area A enclosed by the graph of $f(x) = \frac{1}{2}x + 3$, the x-axis, and the lines $x = 2$ and $x = 4$ by partitioning the closed interval $[2, 4]$ into four subintervals:

$$\left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right], \left[3, \frac{7}{2}\right], \left[\frac{7}{2}, 4\right].$$

(a) Using the left endpoint of each subinterval, draw four small rectangles that lie below the graph of f and sum the areas of the four rectangles.

(b) Using the right endpoint of each subinterval, draw four small rectangles that lie above the graph of f and sum the areas of the four rectangles.

(c) Compare the answers from parts (a) and (b) to the exact area $A = 9$ and to the estimates obtained in Example 1.

6. Approximate the area A enclosed by the graph of $f(x) = 6 - 2x$, the x-axis, and the lines $x = 1$ and $x = 3$ by partitioning the closed interval $[1, 3]$ into four subintervals:

$$\left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right].$$

(a) Using the right endpoint of each subinterval, draw four small rectangles that lie below the graph of f and sum the areas of the four rectangles.

(b) Using the left endpoint of each subinterval, draw four small rectangles that lie above the graph of f and sum the areas of the four rectangles.

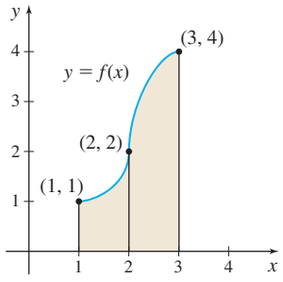
(c) Compare the answers from parts (a) and (b) to the exact area $A = 4$.

In Problems 7 and 8, refer to the graphs below. Approximate the shaded area under the graph of f:

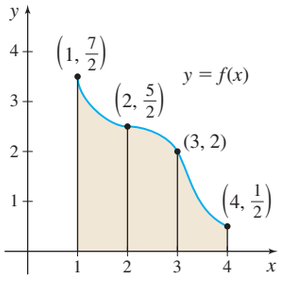
(a) *By constructing rectangles using the left endpoint of each subinterval.*

(b) *By constructing rectangles using the right endpoint of each subinterval.*

7.



8.



In Problems 9–12, partition each interval into n subintervals each of the same length.

9. $[1, 4]$ with $n = 3$

10. $[0, 9]$ with $n = 9$

11. $[-1, 4]$ with $n = 10$

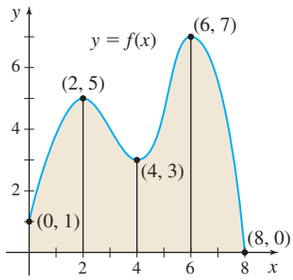
12. $[-4, 4]$ with $n = 16$

In Problems 13 and 14, refer to the graphs. Approximate the shaded area:

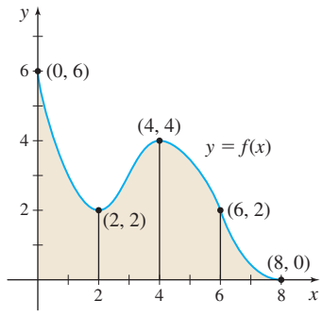
(a) *By using lower sums s_n (rectangles that lie below the graph of f).*

(b) *By using upper sums S_n (rectangles that lie above the graph of f).*

13.



14.



15. Area Under a Graph Consider the area under the graph of $y = x$ from 0 to 3.

(a) Sketch the graph and the area under the graph.

(b) Partition the interval $[0, 3]$ into n subintervals each of equal length.

(c) Show that $s_n = \sum_{i=1}^{n} (i - 1)\left(\frac{3}{n}\right)^2$.

(d) Show that $S_n = \sum_{i=1}^{n} i\left(\frac{3}{n}\right)^2$.

(e) Show that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = \frac{9}{2}$.

16. Area Under a Graph Consider the area under the graph of $y = 4x$ from 0 to 5.

(a) Sketch the graph and the corresponding area.

(b) Partition the interval $[0, 5]$ into n subintervals each of equal length.

(c) Show that $s_n = \sum_{i=1}^{n} (i - 1)\frac{100}{n^2}$.

(d) Show that $S_n = \sum_{i=1}^{n} i\frac{100}{n^2}$.

(e) Show that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = 50$.

In Problems 17–22, approximate the area A under the graph of each function f from a to b for $n = 4$ and $n = 8$ subintervals:

(a) *By using lower sums s_n (rectangles that lie below the graph of f).*

(b) *By using upper sums S_n (rectangles that lie above the graph of f).*

17. $f(x) = -x + 10$ on $[0, 8]$

18. $f(x) = 2x + 5$ on $[2, 6]$

19. $f(x) = 16 - x^2$ on $[0, 4]$

20. $f(x) = x^3$ on $[0, 8]$

21. $f(x) = \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **22.** $f(x) = \sin x$ on $[0, \pi]$

23. Rework Example 3 by using lower sums s_n (rectangles that lie below the graph of f).

24. Rework Example 4 by using upper sums S_n (rectangles that lie above the graph of f).

In Problems 25–32, find the area A under the graph of f from a to b:

(a) *By using lower sums s_n (rectangles that lie below the graph of f).*

(b) *By using upper sums S_n (rectangles that lie above the graph of f).*

(c) *Compare the work required in (a) and (b). Which is easier? Could you have predicted this?*

25. $f(x) = 2x + 1$ from $a = 0$ to $b = 4$

26. $f(x) = 3x + 1$ from $a = 0$ to $b = 4$

27. $f(x) = 12 - 3x$ from $a = 0$ to $b = 4$

28. $f(x) = 5 - x$ from $a = 0$ to $b = 4$

29. $f(x) = 4x^2$ from $a = 0$ to $b = 2$

30. $f(x) = \frac{1}{2}x^2$ from $a = 0$ to $b = 3$

31. $f(x) = 4 - x^2$ from $a = 0$ to $b = 2$

32. $f(x) = 12 - x^2$ from $a = 0$ to $b = 3$

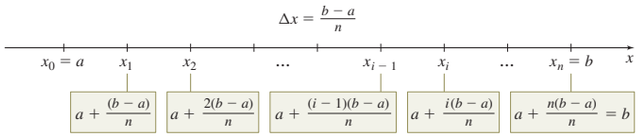
Applications and Extensions

In Problems 33–38, find the area under the graph of f from a to b. [Hint: Partition the closed interval $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$, where $a = x_0 < x_1 < \cdots < x_i < \cdots < x_{n-1} < x_n = b$, and each subinterval is of length $\Delta x = \frac{b-a}{n}$. As the figure below illustrates, the endpoints of each subinterval, written in terms of n, are

$$x_0 = a, \quad x_1 = a + \frac{b-a}{n}, \quad x_2 = a + 2\left(\frac{b-a}{n}\right), \ldots,$$

$$x_{i-1} = a + (i-1)\left(\frac{b-a}{n}\right), \quad x_i = a + i\left(\frac{b-a}{n}\right), \ldots,$$

$$x_n = a + n\left(\frac{b-a}{n}\right)$$



33. $f(x) = x + 3$ from $a = 1$ to $b = 3$

34. $f(x) = 3 - x$ from $a = 1$ to $b = 3$

35. $f(x) = 2x + 5$ from $a = -1$ to $b = 2$

36. $f(x) = 2 - 3x$ from $a = -2$ to $b = 0$

37. $f(x) = 2x^2 + 1$ from $a = 1$ to $b = 3$

38. $f(x) = 4 - x^2$ from $a = 1$ to $b = 2$

In Problems 39–42, approximate the area A under the graph of each function f by partitioning $[a, b]$ into 20 subintervals of equal length and using an upper sum.

39. $f(x) = xe^x$ on $[0, 8]$ **40.** $f(x) = \ln x$ on $[1, 3]$

41. $f(x) = \frac{1}{x}$ on $[1, 5]$ **42.** $f(x) = \frac{1}{x^2}$ on $[2, 6]$

43. **(a)** Graph $y = \frac{4}{x}$ from $x = 1$ to $x = 4$ and shade the area under its graph.

(b) Partition the interval $[1, 4]$ into n subintervals of equal length.

(c) Show that the lower sum s_n is $s_n = \sum_{i=1}^{n} \frac{4}{\left(1 + \frac{3i}{n}\right)}\left(\frac{3}{n}\right)$.

(d) Show that the upper sum S_n is

$$S_n = \sum_{i=1}^{n} \frac{4}{\left(1 + \frac{3(i-1)}{n}\right)}\left(\frac{3}{n}\right)$$

(e) Complete the following table:

n	5	10	50	100
s_n				
S_n				

(f) Use the table to give an upper and lower bound for the area.

Challenge Problems

44. **Area Under a Graph** Approximate the area under the graph of $f(x) = x$ from $a \geq 0$ to b by using lower sums s_n and upper sums S_n for a partition of $[a, b]$ into n subintervals, each of length $\frac{b-a}{n}$. Show that

$$s_n < \frac{b^2 - a^2}{2} < S_n$$

45. **Area Under a Graph** Approximate the area under the graph of $f(x) = x^2$ from $a \geq 0$ to b by using lower sums s_n and upper sums S_n for a partition of $[a, b]$ into n subintervals, each of length $\frac{b-a}{n}$. Show that

$$s_n < \frac{b^3 - a^3}{3} < S_n$$

46. **Area of a Right Triangle** Use lower sums s_n (rectangles that lie inside the triangle) and upper sums S_n (rectangles that lie outside the triangle) to find the area of a right triangle of height H and base B.

47. **Area of a Trapezoid** Use lower sums s_n (rectangles that lie inside the trapezoid) and upper sums S_n (rectangles that lie outside the trapezoid) to find the area of a trapezoid of heights H_1 and H_2 and base B.

5.2 The Definite Integral

OBJECTIVES *When you finish this section, you should be able to:*

1 **Define a definite integral as the limit of Riemann sums (p. 11)**

2 **Find a definite integral using the limit of Riemann sums (p. 14)**

ORIGINS Riemann sums are named after the German mathematician Georg Friedrich Bernhard Riemann (1826–1866). Early in his life, Riemann was home schooled. At age 14, he was sent to a lyceum (high school) and then the University of Göttingen to study theology. Once at Göttingen, he asked for and received permission from his father to study mathematics. He completed his PhD under Karl Friedrich Gauss (1777–1855). In his thesis, Riemann used topology to analyze complex functions. Later he developed a theory of geometry to describe real space. His ideas were far ahead of their time and were not truly appreciated until they provided the mathematical framework for Einstein's Theory of Relativity.

The area A under the graph of $y = f(x)$ from a *to* b is obtained by finding

$$A = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(C_i)\Delta x \tag{1}$$

where the following assumptions were made:

- The function f is continuous on $[a, b]$.
- The function f is nonnegative on $[a, b]$.
- The closed interval $[a, b]$ is partitioned into n subintervals, each of length

$$\Delta x = \frac{b-a}{n}$$

- $f(c_i)$ is the absolute minimum value of f on the ith subinterval, $i = 1, 2, \ldots, n$.
- $f(C_i)$ is the absolute maximum value of f on the ith subinterval, $i = 1, 2, \ldots, n$.

In Section 5.1, we found the area A under the graph of f from a to b by choosing either the number c_i, where f has an absolute minimum on the ith subinterval, or the number C_i, where f has an absolute maximum on the ith subintervals. Suppose we arbitrarily choose a number u_i in each subinterval $[x_{i-1}, x_i]$, and draw rectangles of height $f(u_i)$ and width Δx. Then from the definitions of absolute minimum value and absolute maximum value

$$f(c_i) \le f(u_i) \le f(C_i)$$

and, since $\Delta x > 0$,

$$f(c_i)\Delta x \le f(u_i)\Delta x \le f(C_i)\Delta x$$

Then

$$\sum_{i=1}^{n} f(c_i)\Delta x \le \sum_{i=1}^{n} f(u_i)\Delta x \le \sum_{i=1}^{n} f(C_i)\Delta x$$

$$s_n \le \sum_{i=1}^{n} f(u_i)\Delta x \le S_n$$

Since $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = A$, by the Squeeze Theorem, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\Delta x = A$$

In other words, we can use any number u_i in the ith subinterval to find the area A.

NEED TO REVIEW? The Squeeze Theorem is discussed in Section 1.4, pp. xx–xx.

1 Define a Definite Integral as the Limit of Riemann Sums

We now investigate sums of the form

$$\sum_{i=1}^{n} f(u_i)\Delta x_i$$

using the following more general assumptions:

- The function f is not necessarily continuous on $[a, b]$.
- The function f is not necessarily nonnegative on $[a, b]$.

- The lengths $\Delta x_i = x_i - x_{i-1}$ of the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$ of $[a, b]$ are not necessarily equal.
- The number u_i may be any number in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$.

The sums $\sum_{i=1}^{n} f(u_i)\Delta x_i$, called **Riemann sums** for f on $[a, b]$, form the foundation of integral calculus.

EXAMPLE 1 Forming Riemann Sums

For the function $f(x) = x^2 - 3$, $0 \le x \le 6$, partition the interval $[0, 6]$ into 4 subintervals $[0, 1]$, $[1, 2]$, $[2, 4]$, $[4, 6]$ and form the Riemann sum for which

(a) u_i is the left endpoint of each subinterval.

(b) u_i is the midpoint of each subinterval.

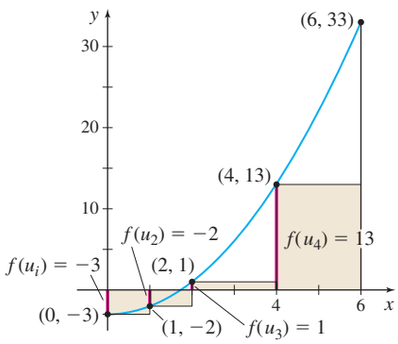


Figure 16 $f(x) = x^2 - 3$, $0 \le x \le 6$

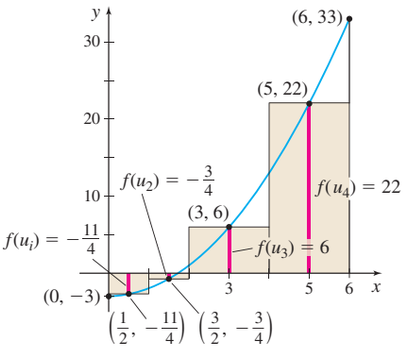


Figure 17 $f(x) = x^2 - 3$, $0 \le x \le 6$

Solution In forming Riemann sums $\sum_{i=1}^{n} f(u_i)\Delta x_i$, n is the number of subintervals in the partition, $f(u_i)$ is the value of f at the number u_i chosen in the ith subinterval, and Δx_i is the length of the ith subinterval.

(a) Figure 16 shows the graph of f, the partition of the interval $[0, 6]$ into the 4 subintervals, and values of $f(u_i)$ at the left endpoint of each subinterval, namely,

$$f(u_1) = f(0) = -3 \qquad f(u_2) = f(1) = -2$$
$$f(u_3) = f(2) = 1 \qquad f(u_4) = f(4) = 13$$

The 4 subintervals have length

$$\Delta x_1 = 1 - 0 = 1 \quad \Delta x_2 = 2 - 1 = 1 \quad \Delta x_3 = 4 - 2 = 2 \quad \Delta x_4 = 6 - 4 = 2$$

The Riemann sum is formed by adding the products $f(u_i)\Delta x_i$ for $i = 1, 2, 3, 4$

$$\sum_{i=1}^{4} f(u_i)\Delta x_i = -3 \cdot 1 + (-2) \cdot 1 + 1 \cdot 2 + 13 \cdot 2 = 23$$

(b) If u_i is chosen as the midpoint of each subinterval, then the value of $f(u_i)$ at the midpoint of each subinterval is

$$f(u_1) = f\left(\frac{1}{2}\right) = -\frac{11}{4} \qquad f(u_2) = f\left(\frac{3}{2}\right) = -\frac{3}{4}$$
$$f(u_3) = f(3) = 6 \qquad f(u_4) = f(5) = 22$$

The Riemann sum formed by adding the products $f(u_i)\Delta x_i$ for $i = 1, 2, 3, 4$ is

$$\sum_{i=1}^{4} f(u_i)\Delta x_i = -\frac{11}{4} \cdot 1 + \left(-\frac{3}{4}\right) \cdot 1 + 6 \cdot 2 + 22 \cdot 2 = \frac{105}{2} = 52.5$$

See Figure 17. ■

NOW WORK Problem 9.

Suppose a function f is defined on a closed interval $[a, b]$, and we partition the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{i-1}, x_i], \quad \ldots, \quad [x_{n-1}, x_n]$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_{n-1} < x_n = b$$

These subintervals are not necessarily of the same length. Denote the length of the first interval by $\Delta x_1 = x_1 - x_0$, the length of the second interval by $\Delta x_2 = x_2 - x_1$, and so

on. In general, the length of the ith subinterval is

$$\boxed{\Delta x_i = x_i - x_{i-1}}$$

for $i = 1, 2, \ldots, n$. This set of subintervals of the interval $[a, b]$ is called a **partition** of $[a, b]$. The length of the largest subinterval in a partition is called the **norm** of the partition and is denoted by $\max \Delta x_i$.

DEFINITION Definite Integral

Let f be a function defined on the closed interval $[a, b]$. Partition $[a, b]$ into n subintervals of length $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$. Choose a number u_i in each subinterval, evaluate $f(u_i)$, and form the Riemann sums $\sum_{i=1}^{n} f(u_i)\Delta x_i$. If $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(u_i)\Delta x_i = I$ exists and does not depend on the choice of the partition or on the choice of u_i, then the number I is called the **Riemann integral** or **definite integral** of f from a to b and is denoted by the symbol $\int_a^b f(x)\,dx$. That is,

$$\boxed{\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(u_i)\Delta x_i}$$

When the above limit exists, then we say that f is **integrable over** $[a, b]$.

For the definite integral $\int_a^b f(x)\,dx$, the number a is called the **lower limit of integration**, the number b is called the **upper limit of integration**, the symbol $\int$ (an elongated S to remind you of summation) is called the **integral sign**, $f(x)$ is called the **integrand**, and dx is the differential of the independent variable x. The variable used in the definite integral is an *artificial* or a *dummy* variable because it may be replaced by any other symbol. For example,

$$\int_a^b f(x)\,dx \qquad \int_a^b f(t)\,dt \qquad \int_a^b f(s)\,ds \qquad \int_a^b f(\theta)\,d\theta$$

all denote the definite integral of f from a to b, and if any of them exist, they are all equal to the same number.

EXAMPLE 2 Expressing the Limit of Riemann Sums as a Definite Integral

(a) Find the Riemann sums for $f(x) = x^2 - 3$ on the closed interval $[0, 6]$ for a partition of $[0, 6]$ into n subintervals of length $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$.

(b) Assuming that the limit of the Riemann sums exists as $\max \Delta x_i \to 0$, express the limit as a definite integral.

Solution **(a)** The Riemann sums for $f(x) = x^2 - 3$ on the closed interval $[0, 6]$ are

$$\sum_{i=1}^{n} f(u_i)\Delta x_i = \sum_{i=1}^{n} \left(u_i^2 - 3\right)\Delta x_i$$

where $[0, 6]$ is partitioned into n subintervals $[x_{i-1}, x_i]$, and u_i is some number in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$.

(b) Since $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \left(u_i^2 - 3\right)\Delta x_i$ exists, then

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \left(u_i^2 - 3\right)\Delta x_i = \int_0^6 (x^2 - 3)\,dx$$

■

NOW WORK Problem 15.

If f is integrable over $[a, b]$, then $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} f(u_i)\Delta x_i$ exists for any choice of u_i in the ith subinterval, so we are free to choose the u_i any way we please. The choices could be the left endpoint of each subinterval, or the right endpoint, or the midpoint, or any other number in each subinterval. Also, $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} f(u_i)\Delta x_i$ is independent of the partition of the closed interval $[a, b]$, provided $\max \Delta x_i$ can be made as close as we please to 0. It is these flexibilities that make the definite integral so important in engineering, physics, chemistry, geometry, and economics.

In defining the definite integral $\int_a^b f(x)\,dx$, we assumed that $a < b$. To remove this restriction, we give the following definitions.

DEFINITION

If $f(a)$ is defined, then

$$\int_a^a f(x)\,dx = 0$$

If $a > b$ and if $\int_b^a f(x)\,dx$ exists, then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

IN WORDS Interchanging the limits of integration reverses the sign of the definite integral.

For example,

$$\int_1^1 x^2\,dx = 0 \qquad \text{and} \qquad \int_3^2 x^2\,dx = -\int_2^3 x^2\,dx$$

Next we give a condition on the function f that guarantees f is integrable. The proof of this result may be found in advanced calculus texts.

THEOREM Existence of the Definite Integral

If a function f is continuous on a closed interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ exists.

The two conditions of the theorem deserve special attention. First, f is defined on a *closed* interval, and second, f is *continuous* on that interval. There are some functions that are continuous on an open interval (or even a half-open interval) for which the integral does not exist. For example, although $f(x) = \dfrac{1}{x^2}$ is continuous on $(0, 1)$ (and on $(0, 1]$), the definite integral $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ does not exist. Also, there are many examples of discontinuous functions for which the integral exists. (See Problems 64 and 65.)

IN WORDS If a function f is continuous on $[a, b]$, then it is integrable over $[a, b]$. But if f is not continuous on $[a, b]$, then f may or may not be integrable over $[a, b]$.

2 Find a Definite Integral Using the Limit of Riemann Sums

Suppose f is integrable over the closed interval $[a, b]$. To find

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(u_i)\Delta x_i$$

using Riemann sums, we usually partition $[a, b]$ into n subintervals, each of the same length $\Delta x = \dfrac{b-a}{n}$. Such a partition is called a **regular partition**. For a regular partition, the norm of the partition is

$$\max \Delta x_i = \frac{b-a}{n}$$

Since $\lim_{n\to\infty} \frac{b-a}{n} = 0$, it follows that for a regular partition, the two statements

$$\max \Delta x_i \to 0 \qquad \text{and} \qquad n \to \infty$$

are interchangeable. As a result, for regular partitions $\Delta x = \frac{b-a}{n}$ and

$$\boxed{\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(u_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\Delta x}$$

The next result uses Riemann sums to establish a formula to find the definite integral of a constant function.

THEOREM

If $f(x) = h$, where h is some constant, then

$$\boxed{\int_a^b f(x)\,dx = \int_a^b h\,dx = h(b-a)}$$

Proof The constant function $f(x) = h$ is continuous on the set of real numbers and so is integrable. We form the Riemann sums for f on the closed interval $[a, b]$ using a regular partition. Then $\Delta x_i = \Delta x = \frac{b-a}{n}$, $i = 1, 2, \ldots, n$. The Riemann sums of f on the interval $[a, b]$ are

$$\sum_{i=1}^{n} f(u_i)\Delta x_i = \sum_{i=1}^{n} f(u_i)\Delta x = \sum_{i=1}^{n} h\,\Delta x = \sum_{i=1}^{n} h\left(\frac{b-a}{n}\right)$$

($f(x) = h$)

$$= h\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} 1 = h\left(\frac{b-a}{n}\right)\cdot n = h(b-a)$$

Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\Delta x = \lim_{n\to\infty} [h(b-a)] = h(b-a)$$

So,

$$\int_a^b h\,dx = h(b-a)$$ ■

For example,

$$\int_1^2 3\,dx = 3(2-1) = 3 \qquad \int_2^6 dx = 1(6-2) = 4 \qquad \int_{-3}^4 (-2)\,dx = (-2)[4-(-3)] = -14$$

NOW WORK **Problem 23.**

EXAMPLE 3 Finding a Definite Integral Using the Limit of Riemann Sums

Find $\int_0^3 (3x-8)\,dx$.

Solution Since the integrand $f(x) = 3x - 8$ is continuous on the closed interval $[0, 3]$, the function f is integrable over $[0, 3]$. Although we can use any partition of $[0, 3]$ whose norm can be made as close to 0 as we please, and we can choose any u_i in each subinterval, we use a regular partition and choose u_i as the right endpoint of each subinterval. This will result in a simple expression for the Riemann sums.

Partition $[0, 3]$ into n subintervals, each of length $\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$. The endpoints of each subinterval of the partition, written in terms of n, are

$$x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = 2\left(\frac{3}{n}\right), \ldots, x_{i-1} = (i-1)\left(\frac{3}{n}\right),$$

$$x_i = i\left(\frac{3}{n}\right), \ldots, x_n = n\left(\frac{3}{n}\right) = 3$$

The Riemann sums of $f(x) = 3x - 8$ from 0 to 3, using $u_i = x_i = \dfrac{3i}{n}$ (the right endpoint) and $\Delta x = \dfrac{3}{n}$, are

$$\sum_{i=1}^{n} f(u_i)\,\Delta x_i = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n}(3x_i - 8)\frac{3}{n} = \sum_{i=1}^{n}\left[3\left(\frac{3i}{n}\right) - 8\right]\frac{3}{n}$$

($\Delta x = \frac{3}{n}$; $x_i = \frac{3i}{n}$)

$$= \sum_{i=1}^{n}\left(\frac{27i}{n^2} - \frac{24}{n}\right) = \frac{27}{n^2}\sum_{i=1}^{n} i - \frac{24}{n}\sum_{i=1}^{n} 1$$

$$= \frac{27}{n^2}\cdot\frac{n(n+1)}{2} - \frac{24}{n}\cdot n = \frac{27}{2} + \frac{27}{2n} - 24 = -\frac{21}{2} + \frac{27}{2n}$$

Now

$$\int_0^3 (3x - 8)\,dx = \lim_{n\to\infty}\left(-\frac{21}{2} + \frac{27}{2n}\right) = -\frac{21}{2}$$ ■

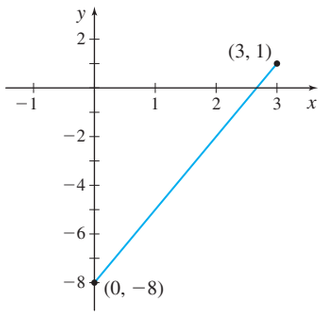


Figure 18 $f(x) = 3x - 8,\ 0 \le x \le 3$

NOW WORK **Problem 45.**

Figure 18 shows the graph of $f(x) = 3x - 8$ on $[0, 3]$. Since f is not nonnegative on $[0, 3]$, we cannot interpret $\int_0^3 (3x - 8)\,dx$ as an area. The fact that the answer is negative is further evidence that this is not an area problem. When finding a definite integral, do not presume it represents area. As you will see in Section 5.4 and in Chapter 6, the definite integral has many interpretations. Interestingly enough, definite integrals are used to find the volume of a solid of revolution, the length of a graph, the work done by a variable force, and other quantities.

EXAMPLE 4 Interpreting a Definite Integral

Determine if each definite integral can be interpreted as an area. If it can, describe the area; if it cannot, explain why.

(a) $\displaystyle\int_0^{3\pi/4} \cos x\,dx$ **(b)** $\displaystyle\int_2^{10} |x - 4|\,dx$

Solution **(a)** See Figure 19 on page 17. Since $\cos x < 0$ on the interval $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right)$, the integral $\int_0^{3\pi/4} \cos x\,dx$ cannot be interpreted as area.

(b) See Figure 20. Since $|x - 4| \ge 0$ on the interval $[2, 10]$, the integral $\int_2^{10} |x - 4|\,dx$ can be interpreted as the area enclosed by the graph of $y = |x - 4|$, the x-axis, and the lines $x = 2$ and $x = 10$.

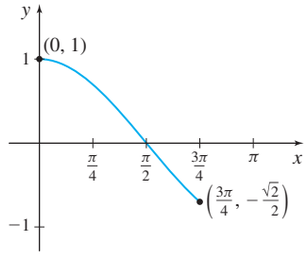


Figure 19 $f(x) = \cos x, 0 \le x \le \dfrac{3\pi}{4}$

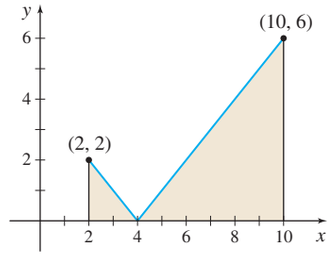


Figure 20 $f(x) = |x - 4|, 2 \le x \le 10$ ■

NOW WORK **Problem 33.**

EXAMPLE 5 Finding a Definite Integral Using Technology

(a) Use a graphing utility to find $\int_{1/2}^{4} \ln x \, dx$

(b) Use a computer algebra system to find $\int_{1/2}^{4} \ln x \, dx$.

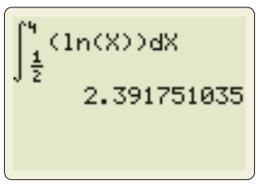


Figure 21

Solution **(a)** A graphing utility, such as a TI-84, provides only an approximate numerical answer to the integral $\int_{1/2}^{4} \ln x \, dx$. As shown in Figure 21, $\int_{1/2}^{4} \ln x \, dx \approx 2.391751035$.

(b) Because a computer algebra system manipulates symbolically, it can find an exact value of the definite integral. Using MuPAD in Scientific WorkPlace, we get

$$\int_{1/2}^{4} \ln x \, dx = \frac{17}{2} \ln 2 - \frac{7}{2}$$

An approximate numerical value of the definite integral using MuPAD is 2.3918. ■

NOW WORK **Problem 43.**

5.2 Assess Your Understanding

Concepts and Vocabulary

1. If an interval $[a, b]$ is partitioned into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\ldots$, $[x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, then the set of subintervals of the interval $[a, b]$ is called a(n) _____ of $[a, b]$.

2. *Multiple Choice* In a regular partition of [0, 40] into 20 subintervals, $\Delta x =$ [**(a)** 20 **(b)** 40 **(c)** 2 **(d)** 4].

3. *True or False* A function f defined on the closed interval $[a, b]$ has an infinite number of Riemann sums.

4. In the notation for a definite integral $\int_a^b f(x)\,dx$, a is called the _____ _____, b is called the _____ _____, $\int$ is called the _____ _____, and $f(x)$ is called the _____.

5. If $f(a)$ is defined, $\int_a^a f(x)\,dx =$ _____.

6. *True or False* If a function f is integrable over a closed interval $[a, b]$, then $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$.

7. *True or False* If a function f is continuous on a closed interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ exists.

8. *Multiple Choice* Since $\int_0^2 (3x - 8)\,dx = -10$, then $\int_2^0 (3x - 8)\,dx =$ [**(a)** -2 **(b)** 10 **(c)** 5 **(d)** 0].

Skill Building

In Problems 9–12, find the Riemann sum for each function f for the partition and the numbers u_i listed.

9. $f(x) = x, 0 \le x \le 2$. Partition the interval [0, 2] as follows:

$$x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1, x_5 = 2;$$

$$\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right], [1, 2]$$

and choose

$$u_1 = \frac{1}{8}, u_2 = \frac{3}{8}, u_3 = \frac{5}{8}, u_4 = \frac{7}{8}, u_5 = \frac{9}{8}.$$

10. $f(x) = x, 0 \le x \le 2$. Partition the interval [0, 2] as follows: $\left[0, \frac{1}{2}\right]$, [1, 0], $\left[1, \frac{3}{2}\right]$, $\left[\frac{3}{2}, 2\right]$, and choose $u_1 = \frac{1}{2}$, $u_2 = 1$, $u_3 = \frac{3}{2}$, $u_4 = 2$.

11. $f(x) = x^2, -2 \le x \le 1$. Partition the interval $[-2, 1]$ as follows: $[-2, -1]$, $[-1, 0]$, $[0, 1]$ and choose $u_1 = -\frac{3}{2}$, $u_2 = -\frac{1}{2}$, $u_3 = \frac{1}{2}$.

12. $f(x) = x^2$, $1 \le x \le 2$. Partition the interval $[1, 2]$ as follows: $\left[1, \frac{5}{4}\right], \left[\frac{5}{4}, \frac{3}{2}\right], \left[\frac{3}{2}, \frac{7}{4}\right], \left[\frac{7}{4}, 2\right]$ and choose $u_1 = \frac{5}{4}$, $u_2 = \frac{3}{2}$, $u_3 = \frac{7}{4}$, $u_4 = 2$.

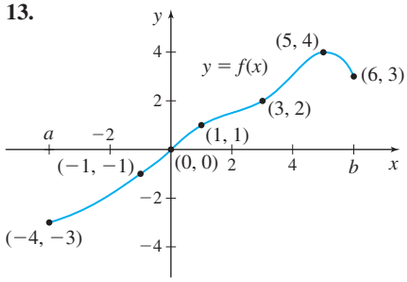
In Problems 13 and 14, the graph of a function f defined on an interval $[a, b]$ is given. (Answers will vary.)

(a) *Partition the interval $[a, b]$ into five subintervals (not necessarily of the same size).*

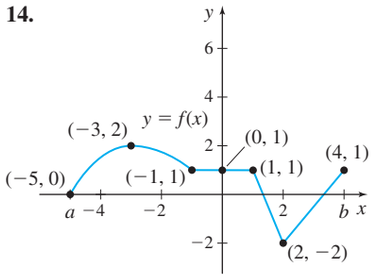
(b) *Approximate $\int_a^b f(x)\,dx$ by choosing u_i as the left endpoint of each subinterval and using Riemann sums.*

(c) *Approximate $\int_a^b f(x)\,dx$ by choosing u_i as the right endpoint of each subinterval and using Riemann sums.*

13.



14.



In Problems 15–22, write the limit of the Riemann sums as a definite integral.

15. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} \left(e^{u_i} + 2\right) \Delta x_i$ on $[0, 2]$

16. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} \ln u_i \, \Delta x_i$ on $[1, 8]$

17. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} \cos u_i \, \Delta x_i$ on $[0, 2\pi]$

18. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} (\cos u_i + \sin u_i) \, \Delta x_i$ on $[0, \pi]$

19. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} \frac{2}{u_i^2} \Delta x_i$ on $[1, 4]$

20. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} u_i^{1/3} \Delta x_i$ on $[0, 8]$

21. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} u_i \ln u_i \, \Delta x_i$ on $[1, e]$

22. $\lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} \ln(u_i + 1) \Delta x_i$ on $[0, e]$

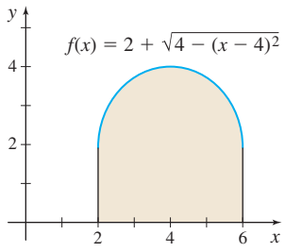
In Problems 23–28, find each definite integral.

23. $\int_{-3}^{4} e\,dx$ **24.** $\int_{0}^{3} (-\pi)\,dx$ **25.** $\int_{3}^{0} (-\pi)\,dt$

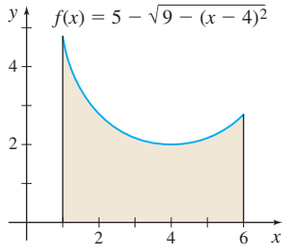
26. $\int_{7}^{2} 2\,ds$ **27.** $\int_{4}^{4} 2\theta\,d\theta$ **28.** $\int_{-1}^{-1} 8\,dr$

In Problems 29–32, the graph of a function is shown. Express the shaded area as a definite integral.

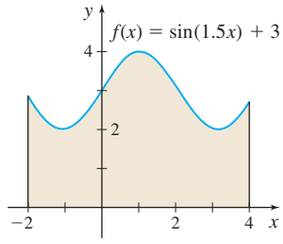
29.



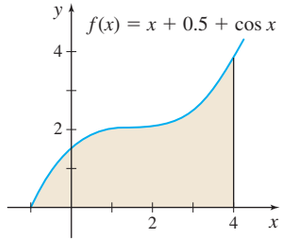
30.



31.



32.



In Problems 33–38, determine which of the following definite integrals can be interpreted as area. For those that can, describe the area; for those that cannot, explain why.

33. $\int_{0}^{\pi} \sin x\,dx$ **34.** $\int_{-\pi/4}^{\pi/4} \tan x\,dx$

35. $\int_{1}^{4} (x-2)^{1/3} dx$ **36.** $\int_{1}^{4} (x+2)^{1/3} dx$

37. $\int_{1}^{4} [|x| - 2]\,dx$ **38.** $\int_{-2}^{4} |x|\,dx$

In Problems 39–44:

(a) *For each function defined on the given interval, use a regular partition to form Riemann sums.*

(b) *Express the limit as $n \to \infty$ of the Riemann sums as a definite integral.*

CAS ***(c)*** *Use a computer algebra system to find the value of the definite integral in (b).*

39. $f(x) = x^2 - 1$ on $[0, 2]$ **40.** $f(x) = x^3 - 2$ on $[0, 5]$

41. $f(x) = \sqrt{x+1}$ on $[0, 3]$ **42.** $f(x) = \sin x$ on $[0, \pi]$

43. $f(x) = e^x$ on $[0, 2]$ **44.** $f(x) = e^{-x}$ on $[0, 1]$

In Problems 45 and 46, find each definite integral using Riemann sums.

45. $\int_0^1 (x-4)\,dx$ **46.** $\int_0^3 (3x-1)\,dx$

CAS *In Problems 47–50, for each function defined on the interval $[a, b]$:*

(a) *Complete the table of Riemann sums using a regular partition of $[a, b]$.*

n	10	50	100
Using left endpoints			
Using right endpoints			
Using the midpoint			

(b) *Use a CAS to find the definite integral.*

(c) *Compare the answers in (a) and (b). Which Riemann sum gives the best approximation to the definite integral?*

47. $f(x) = 2 + \sqrt{x}$ on $[1, 5]$ **48.** $f(x) = e^x + e^{-x}$ on $[-1, 3]$

49. $f(x) = \dfrac{3}{1+x^2}$ on $[-1, 1]$ **50.** $f(x) = \dfrac{1}{\sqrt{x^2+4}}$ on $[0, 2]$

Applications and Extensions

51. Find an approximate value of $\int_1^2 \frac{1}{x}\,dx$ by finding Riemann sums corresponding to a partition of $[1, 2]$ into four subintervals, each of the same length, and evaluating the integrand at the midpoint of each subinterval. Compare your answer with the true value, $0.6931\ldots$.

52. **(a)** Find the approximate value of $\int_0^2 \sqrt{4-x^2}\,dx$ by finding Riemann sums corresponding to a partition of $[0, 2]$ into 16 subintervals, each of the same length, and evaluating the integrand at the left endpoint of each subinterval.

(b) Can $\int_0^2 \sqrt{4-x^2}\,dx$ be interpreted as area? If it can, describe the area; if it cannot, explain why.

(c) Find the actual value of $\int_0^2 \sqrt{4-x^2}dx$ by graphing $y = \sqrt{4-x^2}$ and using a familiar formula from geometry.

53. Units of an Integral In the definite integral $\int_0^5 F(x)\,dx$, F represents a force measured in newtons and x, $0 \le x \le 5$, is measured in meters. What are the units of $\int_0^5 F(x)\,dx$?

54. Units of an Integral In the definite integral $\int_0^{50} C(x)\,dx$, C represents the concentration of a drug in grams per liter and x, $0 \le x \le 50$, is measured in liters of alcohol. What are the units of $\int_0^{50} C(x)\,dx$?

55. Units of an Integral In the definite integral $\int_a^b v(t)\,dt$, v represents velocity measured in meters per second and time t is measured in seconds. What are the units of $\int_a^b v(t)\,dt$?

56. Units of an Integral In the definite integral $\int_a^b S(t)\,dt$, S represents the rate of sales of a corporation measured in millions of dollars per year and time t is measured in years. What are the units of $\int_a^b S(t)\,dt$?

CAS **57. Area**

(a) Graph the function $f(x) = 3 - \sqrt{6x - x^2}$.

(b) Find the area under the graph of f from 0 to 6.

(c) Confirm the answer to (b) using geometry.

CAS **58. Area**

(a) Graph the function $f(x) = \sqrt{4x - x^2} + 2$.

(b) Find the area under the graph of f from 0 to 4.

(c) Confirm the answer to (b) using geometry.

59. The interval $[1, 5]$ is partitioned into eight subintervals each of the same length.

(a) What is the largest Riemann sum of $f(x) = x^2$ that can be found using this partition?

(b) What is the smallest Riemann sum?

(c) Compute the average of these sums.

(d) What integral has been approximated, and what is the integral's exact value?

Challenge Problems

60. The floor function $f(x) = \lfloor x \rfloor$ is not continuous on $[0, 4]$. Show that $\int_0^4 f(x)\,dx$ exists.

61. Consider the **Dirichlet function** f, where

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that $\int_0^1 f(x)\,dx$ does not exist. (*Hint*: Evaluate the Riemann sums in two different ways: first by using rational numbers for u_i and then by using irrational numbers for u_i.)

62. It can be shown (with a certain amount of work) that if $f(x)$ is integrable on the interval $[a, b]$, then so is $|f(x)|$. Is the converse true?

63. If only regular partitions are allowed, then we could not always partition an interval $[a, b]$ in a way that automatically partitions subintervals $[a, c]$ and $[c, b]$ for $a < c < b$. Why not?

64. If f is a function that is continuous on a closed interval $[a, b]$, except at $x_1, x_2, \ldots, x_n$, $n \ge 1$ an integer, where it has a jump discontinuity, show that f is integrable on $[a, b]$.

65. If f is a function that is continuous on a closed interval $[a, b]$, except at $x_1, x_2, \ldots, x_n$, $n \ge 1$ an integer, where it has a removable discontinuity, show that f is integrable on $[a, b]$.

5.3 The Fundamental Theorem of Calculus

OBJECTIVES *When you finish this section, you should be able to:*

1. Use Part 1 of the Fundamental Theorem of Calculus (p. 21)
2. Use Part 2 of the Fundamental Theorem of Calculus (p. 23)
3. Interpret an integral using Part 2 of the Fundamental Theorem of Calculus (p. 23)

In this section, we discuss the Fundamental Theorem of Calculus, a method for finding integrals more easily, avoiding the need to find the limit of Riemann sums. The Fundamental Theorem is aptly named because it links the two branches of calculus: differential calculus and integral calculus. As it turns out, the Fundamental Theorem of Calculus has two parts, each of which relates an integral to an antiderivative.

Suppose f is a function that is continuous on a closed interval $[a, b]$. Then the definite integral $\int_a^b f(x)\,dx$ exists and is equal to a real number. Now if x denotes any number in $[a, b]$, the definite integral $\int_a^x f(t)\,dt$ exists and depends on x. That is, $\int_a^x f(t)\,dt$ is a function of x, which we name I, for "integral."

$$I(x) = \int_a^x f(t)\,dt$$

The domain of I is the closed interval $[a, b]$. The integral that defines I has a *variable upper limit of integration* x. The t that appears in the integrand is a dummy variable. Surprisingly, when we differentiate I with respect to x, we get back the original function f. That is, $\int_a^x f(t)\,dt$ *is an antiderivative of* f.

NEED TO REVIEW? Antiderivatives are discussed in Section 4.8, pp. xx–xx.

THEOREM Fundamental Theorem of Calculus, Part 1

Let f be a function that is continuous on a closed interval $[a, b]$. The function I defined by

$$\boxed{I(x) = \int_a^x f(t)\,dt}$$

has the properties that it is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$\boxed{I'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)}$$

for all x in (a, b).

The proof of Part 1 of the Fundamental Theorem of Calculus is given in Appendix B. However, if the integral $\int_a^x f(t)\,dt$ represents area, we can interpret the theorem using geometry.

Figure 22 shows the graph of a function f that is nonnegative and continuous on a closed interval $[a, b]$. Then $I(x) = \int_a^x f(t)\,dt$ equals the area under the graph of f from a to x.

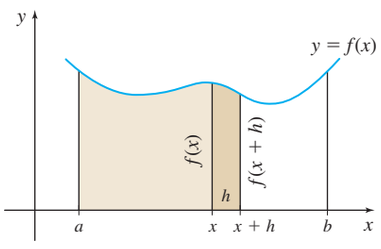


Figure 22

$$I(x) = \int_a^x f(t)\,dt = \text{the area under the graph of } f \text{ from } a \text{ to } x$$

$$I(x+h) = \int_a^{x+h} f(t)\,dt = \text{the area under the graph of } f \text{ from } a \text{ to } x+h$$

$$I(x+h) - I(x) = \text{the area under the graph of } f \text{ from } x \text{ to } x+h$$

$$\frac{I(x+h) - I(x)}{h} = \frac{\text{the area under the graph of } f \text{ from } x \text{ to } x+h}{h} \qquad (1)$$

Based on the definition of a derivative,

$$\lim_{h\to 0}\left[\frac{I(x+h)-I(x)}{h}\right]=I'(x) \tag{2}$$

Since f is continuous, $\lim_{h\to 0} f(x+h)=f(x)$. As $h\to 0$, the area under the graph of f from x to $x+h$ gets closer to the area of a rectangle with width h and height $f(x)$. That is,

$$\lim_{h\to 0}\frac{\text{the area under the graph of } f \text{ from } x \text{ to } x+h}{h}=\lim_{h\to 0}\frac{h[f(x)]}{h}=f(x) \tag{3}$$

Combining (1), (2), and (3), it follows that $I'(x)=f(x)$.

1 Use Part 1 of the Fundamental Theorem of Calculus

EXAMPLE 1 Using Part 1 of the Fundamental Theorem of Calculus

(a) $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{t+1}\,dt=\sqrt{x+1}$ **(b)** $\dfrac{d}{dx}\displaystyle\int_2^x \frac{s^3-1}{2s^2+s+1}ds=\frac{x^3-1}{2x^2+x+1}$ ■

NOW WORK Problem 5.

EXAMPLE 2 Using Part 1 of the Fundamental Theorem of Calculus

Find $\dfrac{d}{dx}\displaystyle\int_4^{3x^2+1}\sqrt{e^t+t}\,dt$.

NEED TO REVIEW? The Chain Rule is discussed in Section 3.1, pp. xx–xx.

Solution The upper limit of integration is a function of x, so we use the Chain Rule along with Part 1 of the Fundamental Theorem of Calculus.

Let $y=\int_4^{3x^2+1}\sqrt{e^t+t}\,dt$ and $u(x)=3x^2+1$. Then $y=\int_4^u \sqrt{e^t+t}\,dt$ and

$$\frac{d}{dx}\int_4^{3x^2+1}\sqrt{e^t+t}\,dt=\frac{dy}{dx}=\underset{\text{Chain Rule}}{\frac{dy}{du}}\cdot\frac{du}{dx}=\left[\frac{d}{du}\int_4^u\sqrt{e^t+t}\,dt\right]\cdot\frac{du}{dx}$$

$$=\underset{\substack{\text{Apply the Fundamental}\\ \text{Theorem}}}{\sqrt{e^u+u}}\cdot\frac{du}{dx}=\sqrt{e^{(3x^2+1)}+3x^2+1}\cdot\underset{u=3x^2+1;\ \frac{du}{dx}=6x}{6x}$$ ■

NOW WORK Problem 11.

EXAMPLE 3 Using Part 1 of the Fundamental Theorem of Calculus

Find $\dfrac{d}{dx}\displaystyle\int_{x^3}^5 (t^4+1)^{1/3}\,dt$.

Solution To use Part 1 of the Fundamental Theorem of Calculus, the variable must be part of the upper limit of integration. So, we use the fact that $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$ to interchange the limits of integration.

$$\frac{d}{dx}\int_{x^3}^5 (t^4+1)^{1/3}\,dt=\frac{d}{dx}\left[-\int_5^{x^3}(t^4+1)^{1/3}\,dt\right]=-\frac{d}{dx}\int_5^{x^3}(t^4+1)^{1/3}\,dt$$

Now we use the Chain Rule. We let $y = \int_5^{x^3} (t^4+1)^{1/3} dt$ and $u(x) = x^3$.

$$\frac{d}{dx}\int_{x^3}^{5}(t^4+1)^{1/3}\,dt = -\frac{d}{dx}\int_5^{x^3}(t^4+1)^{1/3}\,dt = -\frac{dy}{dx} = -\frac{dy}{du}\cdot\frac{du}{dx}$$

Chain Rule

$$= -\frac{d}{du}\int_5^{u}(t^4+1)^{1/3}dt \cdot \frac{du}{dx}$$

$$= -(u^4+1)^{1/3}\cdot\frac{du}{dx} \qquad \text{Apply the Fundamental Theorem}$$

$$= -(x^{12}+1)^{1/3}\cdot 3x^2 \qquad u = x^3;\ \frac{du}{dx} = 3x^2$$

$$= -3x^2(x^{12}+1)^{1/3}$$ ■

NOTE In these examples, the differentiation is with respect to the variable that appears in the upper limit of integration and the answer is a function of that variable.

NOW WORK **Problem 15.**

Part 1 of the Fundamental Theorem of Calculus establishes a relationship between the derivative and the definite integral. Part 2 of the Fundamental Theorem of Calculus provides a method for finding a definite integral without using Riemann sums.

THEOREM Fundamental Theorem of Calculus, Part 2

Let f be a function that is continuous on a closed interval $[a, b]$. If F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Proof Let $I(x) = \int_a^x f(t)\,dt$. Then from Part 1 of the Fundamental Theorem of Calculus, $I = I(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$. So,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad a < x < b$$

RECALL Any two antiderivatives of a function differ by a constant.

That is, $\int_a^x f(t)\,dt$ is an antiderivative of f. So, if F is any antiderivative of f, then

$$F(x) = \int_a^x f(t)\,dt + C$$

where C is some constant. Since F is continuous on $[a, b]$, we have

$$F(a) = \int_a^a f(t)\,dt + C, \qquad F(b) = \int_a^b f(t)\,dt + C$$

NOTE For any constant C, $[F(b)+C]-[F(a)+C] = F(b)-F(a)$. In other words, it does not matter which antiderivative of f is chosen when using Part 2 of the Fundamental Theorem of Calculus, since the same answer is obtained for every antiderivative.

Since, $\int_a^a f(t)\,dt = 0$, subtracting $F(a)$ from $F(b)$ gives

$$F(b) - F(a) = \int_a^b f(t)\,dt$$

Since t is a dummy variable, we can replace t by x and the result follows. ■

As an aid in computation, we introduce the notation

$$\boxed{\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)}$$

The notation $[F(x)]_a^b$ also suggests that to compute $\int_a^b f(x)\,dx$, we first find an antiderivative $F(x)$ of $f(x)$. Then we write $[F(x)]_a^b$ to represent $F(b) - F(a)$.

2 Use Part 2 of the Fundamental Theorem of Calculus

EXAMPLE 4 Using Part 2 of the Fundamental Theorem of Calculus

Use Part 2 of the Fundamental Theorem of Calculus to find:

(a) $\int_{-2}^{1} x^2\,dx$ **(b)** $\int_{0}^{\pi/6} \cos x\,dx$ **(c)** $\int_{0}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx$ **(d)** $\int_{1}^{2} \frac{1}{x}\,dx$

Solution **(a)** An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$. By Part 2 of the Fundamental Theorem of Calculus,

$$\int_{-2}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{1} = \frac{1^3}{3} - \frac{(-2)^3}{3} = \frac{1}{3} + \frac{8}{3} = \frac{9}{3} = 3$$

(b) An antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$. By Part 2 of the Fundamental Theorem of Calculus,

$$\int_{0}^{\pi/6} \cos x\,dx = \Big[\sin x\Big]_{0}^{\pi/6} = \sin\frac{\pi}{6} - \sin 0 = \frac{1}{2}$$

(c) An antiderivative of $f(x) = \frac{1}{\sqrt{1-x^2}}$ is $F(x) = \sin^{-1} x$, provided $|x| < 1$. By Part 2 of the Fundamental Theorem of Calculus,

$$\int_{0}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx = \Big[\sin^{-1} x\Big]_{0}^{\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

(d) An antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln|x|$. By Part 2 of the Fundamental Theorem of Calculus,

$$\int_{1}^{2} \frac{1}{x}\,dx = \Big[\ln|x|\Big]_{1}^{2} = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2$$ ■

NOW WORK Problem 25 and 31.

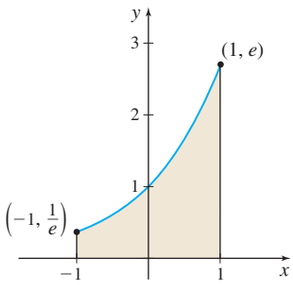


Figure 23 $f(x) = e^x, -1 \le x \le 1$

EXAMPLE 5 Finding the Area Under a Graph

Find the area under the graph of $f(x) = e^x$ from -1 to 1.

Solution Figure 23 shows the graph of $f(x) = e^x$ on the closed interval $[-1, 1]$.

The area A under the graph of f from -1 to 1 is given by

$$A = \int_{-1}^{1} e^x\,dx = \Big[e^x\Big]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e} \approx 2.35$$ ■

NOW WORK Problem 45.

3 Interpret an Integral Using Part 2 of the Fundamental Theorem of Calculus

Part 2 of the Fundamental Theorem of Calculus states that, under certain conditions,

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a) \qquad \text{where } F' = f$$

That is,

$$\int_{a}^{b} F'(x)\,dx = F(b) - F(a)$$

In other words,

> The integral from a to b of the rate of change of F equals the change in F from a to b.

EXAMPLE 6 Interpreting an Integral Whose Integrand Is a Rate of Change

(a) The function $P = P(t)$ relates the population P (in billions of persons) as a function of the time t (in years). Suppose $\int_0^{10} P'(t)\,dt = 3$. Since $P'(t)$ is the rate of change of the population with respect to time, then the change in population from $t = 0$ to $t = 10$ is 3 billion persons.

(b) The function $v = v(t)$ is the speed v (in meters per second) of an object at a time t (in seconds). If $\int_0^5 v(t)\,dt = 8$, then the object travels 8 m during the interval $0 \le t \le 5$. Do you see why? The speed $v = v(t)$ is the rate of change of distance s with respect to time t. That is, $v = \dfrac{ds}{dt}$. ■

This interpretation of an integral is important since it reveals how to go from a rate of change of a function F back to the change itself.

NOW WORK Problem 51.

5.3 Assess Your Understanding

Concepts and Vocabulary

1. According to Part 1 of the Fundamental Theorem of Calculus, if a function f is continuous on a closed interval $[a, b]$, then $\dfrac{d}{dx}\left[\displaystyle\int_a^x f(t)\,dt\right] =$ _____ for all numbers x in the interval.
2. By Part 2 of the Fundamental Theorem of Calculus, $\displaystyle\int_a^b dx =$ _____.
3. *True or False* By Part 2 of the Fundamental Theorem of Calculus, $\displaystyle\int_a^b f(x)\,dx = f(b) - f(a)$.
4. *True or False* $\displaystyle\int_a^b F'(x)\,dx$ can be interpreted as the rate of change in F from a to b.

Skill Building

In Problems 5–18, find each derivative using Part 1 of the Fundamental Theorem of Calculus.

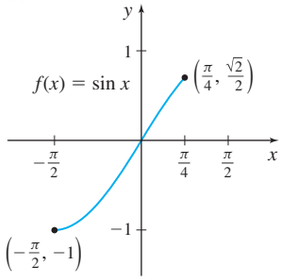
5. $\dfrac{d}{dx}\displaystyle\int_1^x \sqrt{t^2+1}\,dt$
6. $\dfrac{d}{dx}\displaystyle\int_3^x \frac{t+1}{t}\,dt$
7. $\dfrac{d}{dt}\left[\displaystyle\int_0^t (3+x^2)^{3/2}\,dx\right]$
8. $\dfrac{d}{dx}\left[\displaystyle\int_{-4}^x \left(t^3+8\right)^{1/3}\,dt\right]$
9. $\dfrac{d}{dx}\left[\displaystyle\int_1^x \ln u\,du\right]$
10. $\dfrac{d}{dt}\left[\displaystyle\int_4^t e^x\,dx\right]$
11. $\dfrac{d}{dx}\left[\displaystyle\int_1^{2x^3} \sqrt{t^2+1}\,dt\right]$
12. $\dfrac{d}{dx}\left[\displaystyle\int_1^{\sqrt{x}} \sqrt{t^4+5}\,dt\right]$
13. $\dfrac{d}{dx}\left[\displaystyle\int_2^{x^5} \sec t\,dt\right]$
14. $\dfrac{d}{dx}\left[\displaystyle\int_3^{1/x} \sin^5 t\,dt\right]$
15. $\dfrac{d}{dx}\left[\displaystyle\int_x^5 \sin(t^2)\,dt\right]$
16. $\dfrac{d}{dx}\left[\displaystyle\int_x^3 (t^2-5)^{10}\,dt\right]$
17. $\dfrac{d}{dx}\left[\displaystyle\int_{5x^2}^5 (6t)^{2/3}\,dt\right]$
18. $\dfrac{d}{dx}\left[\displaystyle\int_{x^2}^0 e^{10t}\,dt\right]$

In Problems 19–36, use Part 2 of the Fundamental Theorem of Calculus to find each definite integral.

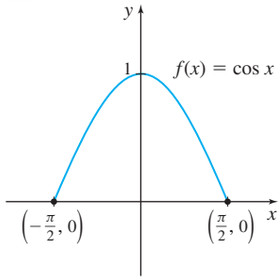
19. $\displaystyle\int_{-2}^3 dx$
20. $\displaystyle\int_{-2}^3 2\,dx$
21. $\displaystyle\int_{-1}^2 x^3\,dx$
22. $\displaystyle\int_1^3 \frac{1}{x^3}\,dx$
23. $\displaystyle\int_0^1 \sqrt{u}\,du$
24. $\displaystyle\int_1^8 \sqrt[3]{y}\,dy$
25. $\displaystyle\int_{\pi/6}^{\pi/2} \csc^2 x\,dx$
26. $\displaystyle\int_0^{\pi/2} \cos x\,dx$
27. $\displaystyle\int_0^{\pi/4} \sec x \tan x\,dx$
28. $\displaystyle\int_{\pi/6}^{\pi/2} \csc x \cot x\,dx$
29. $\displaystyle\int_{-1}^0 e^x\,dx$
30. $\displaystyle\int_{-1}^0 e^{-x}\,dx$
31. $\displaystyle\int_1^e \frac{1}{x}\,dx$
32. $\displaystyle\int_e^1 \frac{1}{x}\,dx$
33. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$
34. $\displaystyle\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx$
35. $\displaystyle\int_{-1}^8 x^{2/3}\,dx$
36. $\displaystyle\int_0^4 x^{3/2}\,dx$

In Problems 37–42, find $\int_a^b f(x)\,dx$ over the domain of f indicated in the graph.

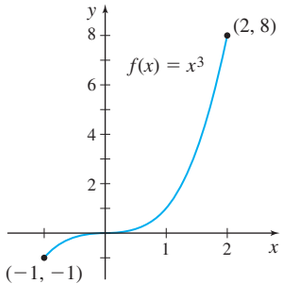
37.



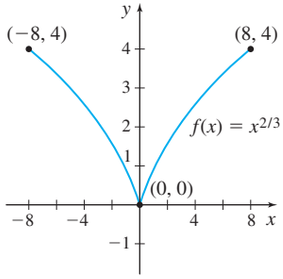
38.



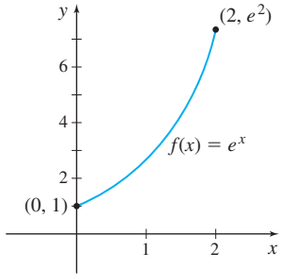
39.



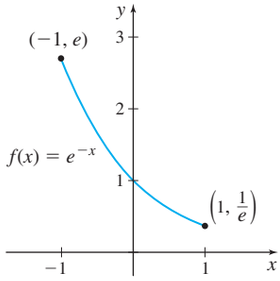
40.



41.



42.



43. Given that $f(x) = (2x^3 - 3)^2$ and $f'(x) = 12x^2(2x^3 - 3)$, find $\int_0^2 [12x^2(2x^3 - 3)]\,dx$.

44. Given that $f(x) = (x^2 + 5)^3$ and $f'(x) = 6x(x^2 + 5)^2$, find $\int_{-1}^2 6x(x^2 + 5)^2\,dx$.

Applications and Extensions

45. Area Find the area under the graph of $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$ from 0 to $\dfrac{1}{2}$.

46. Area Find the area under the graph of $f(x) = \cosh x$ from -1 to 1.

47. Area Find the area under the graph of $f(x) = \dfrac{1}{x^2 + 1}$ from 0 to $\sqrt{3}$.

48. Area Find the area under the graph of $f(x) = \dfrac{1}{1 + x^2}$ from 0 to r, where $r > 0$. What happens as $r \to \infty$?

49. Area Find the area under the graph of $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = r$, where $r > 1$. Then examine the behavior of this area as $r \to \infty$.

50. Area Find the area under the graph of $y = \dfrac{1}{x^2}$ from $x = 1$ to $x = r$, where $r > 1$. Then examine the behavior of this area as $r \to \infty$.

51. Interpreting an Integral The function $R = R(t)$ models the rate of sales of a corporation measured in millions of dollars per year as a function of the time t in years. Interpret the integral $\int_0^2 R(t)\,dt = 23$.

52. Interpreting an Integral The function $v = v(t)$ models the speed v in meters per second of an object at a time t in seconds. Interpret the integral $\int_0^{10} v(t)\,dt = 4.8$.

53. Interpreting an Integral Helium is leaking from a large advertising balloon at a rate of $H(t)$ cubic centimeters per minute, where t is measured in minutes.

(a) Write an integral that models the change in helium in the balloon over the interval $a \le t \le b$.

(b) What are the units of the integral from (a)?

(c) Interpret $\int_0^{300} H(t)\,dt = -100$.

54. Interpreting an Integral Water is being added to a reservoir at a rate of $w(t)$ kiloliters per hour, where t is measured in hours.

(a) Write an integral that models the change in amount of water in the reservoir over the interval $a \le t \le b$.

(b) What are the units of the integral from (a)?

(c) Interpret $\int_0^{36} w(t)\,dt = 800$.

55. Free Fall The speed v of an object dropped from rest is given by $v(t) = 9.8t$, where v is in meters per second and time t is in seconds.

(a) Express the distance traveled in the first 5.2 s as an integral.

(b) Find the distance traveled in 5.2 s.

56. Area Find h so that the area under the graph of $y^2 = x^3$, $0 \le x \le 4$, $y \ge 0$, is equal to the area of a rectangle of base 4 and height h.

57. Area If P is a polynomial that is positive for $x > 0$, and for each $k > 0$ the area under the graph of P from $x = 0$ to $x = k$ is $k^3 + 3k^2 + 6k$, find P.

58. Put It Together If $f(x) = \displaystyle\int_0^x \frac{1}{\sqrt{t^3 + 2}}\,dt$, which of the following is *false?*

(a) f is continuous at x for all $x \ge 0$

(b) $f(1) > 0$

(c) $f(0) = \dfrac{1}{\sqrt{2}}$

(d) $f'(1) = \dfrac{1}{\sqrt{3}}$

In Problems 59–62:

(a) *Use Part of 2 the Fundamental Theorem of Calculus to find each definite integral.*

(b) *Determine whether the integrand is an even function, an odd function, or neither.*

(c) *Make a conjecture about the definite integrals in (a) based on the analysis from (b).*

59. $\int_0^4 x^2 dx$ and $\int_{-4}^4 x^2 dx$

60. $\int_0^4 x^3 dx$ and $\int_{-4}^4 x^3 dx$

61. $\int_0^{\pi/4} \sin x\, dx$ and $\int_{-\pi/4}^{\pi/4} \sin x\, dx$

62. $\int_0^{\pi/4} \sec^2 x\, dx$ and $\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

63. Area Find c, $0 < c < 1$, so that the area under the graph of $y = x^2$ from 0 to c equals the area under the same graph from c to 1.

64. Area Let A be the area under the graph of $y = \dfrac{1}{x}$ from $x = m$ to $x = 2m$, $m > 0$. Which of the following is true about the area A?

(a) A is independent of m.

(b) A increases as m increases.

(c) A decreases as m increases.

(d) A decreases as m increases when $m < \dfrac{1}{2}$ and increases as m increases when $m > \dfrac{1}{2}$.

(e) A increases as m increases when $m < \dfrac{1}{2}$ and decreases as m increases when $m > \dfrac{1}{2}$.

65. Put It Together If F is a function whose derivative is continuous for all real x, find

$$\lim_{h \to 0} \frac{1}{h} \int_c^{c+h} F'(x)\, dx$$

66. Suppose the closed interval $\left[0, \dfrac{\pi}{2}\right]$ is partitioned into n subintervals, each of length Δx, and u_i is an arbitrary number in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Explain why

$$\lim_{n \to \infty} \sum_{i=1}^{n} [(\cos u_i)\, \Delta x] = 1$$

67. The interval $[0, 4]$ is partitioned into n subintervals, each of length Δx, and a number u_i is chosen in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Find $\lim_{n \to \infty} \sum_{i=1}^{n} \left[\sqrt{u_i}\, \Delta x\right]$.

68. If u and v are differentiable functions and f is a continuous function, find a formula for

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\, dt\right]$$

69. Find $\dfrac{d}{dx}\displaystyle\int_x^1 (t-1)^2\, dt$ without integrating. Then check by integrating before differentiating.

70. If f' is continuous on the interval $[a, b]$, show that

$$\int_a^b f(x) f'(x)\, dx = \frac{1}{2}\Big\{[f(b)]^2 - [f(a)]^2\Big\}.$$

[*Hint*: Look at the derivative of $F(x) = \dfrac{[f(x)]^2}{2}$.]

71. If f'' is continuous on the interval $[a, b]$, show that

$$\int_a^b x f''(x)\, dx = b f'(b) - a f'(a) - f(b) + f(a).$$

[*Hint*: Look at the derivative of $F(x) = x f'(x) - f(x)$.]

Challenge Problems

72. What conditions on f and f' guarantee that $f(x) = \int_0^x f'(t)\, dt$?

73. Suppose that F is an antiderivative of f on the interval $[a, b]$. Partition $[a, b]$ into n subintervals, each of length $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$.

(a) Apply the Mean Value Theorem for derivatives to F in each subinterval $[x_{i-1}, x_i]$ to show that there is a point u_i in the subinterval for which $F(x_i) - F(x_{i-1}) = f(u_i)\Delta x_i$.

(b) Show that $\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = F(b) - F(a)$.

(c) Use parts (a) and (b) to explain why

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

(In this alternate proof of Part 2 of the Fundamental Theorem of Calculus, the continuity of f is not assumed.)

74. Given $y = \sqrt{x^2 - 1}(4 - x)$, $1 \le x \le a$, for what number a will $\int_1^a y\, dx$ have a maximum value?

75. Find $a > 0$, so that the area under the graph of $y = x + \dfrac{1}{x}$ from a to $(a + 1)$ is minimum.

76. If n is a known positive integer, for what number c is

$$\int_1^c x^{n-1}\, dx = \frac{1}{n}$$

77. Let $f(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{1 - t^2}}$, $0 < x < 1$.

(a) Find $\dfrac{d}{dx} f(\sin x)$.

(b) Is f one-to-one?

(c) Does f have an inverse?

5.4 Properties of the Definite Integral

OBJECTIVES *When you finish this section, you should be able to:*

1. Use properties of the definite integral (p. 27)
2. Work with the Mean Value Theorem for Integrals (p. 30)
3. Find the average value of a function (p. 31)

We have seen that there are properties of limits that make it easier to find limits, properties of continuity that make it easier to determine continuity, and properties of derivatives that make it easier to find derivatives. Here, we investigate several properties of the definite integral that will make it easier to find integrals.

1 Use Properties of the Definite Integral

NOTE From now on, we will refer to Parts 1 and 2 of the Fundamental Theorem of Calculus simply as the **Fundamental Theorem of Calculus.**

Proofs of most of the properties in this section require the use of the definition of the definite integral. However, if the condition is added that the integrand is continuous, then the Fundamental Theorem of Calculus can be used to establish the properties. We will use this added condition in the proofs included here.

THEOREM The Integral of the Sum of Two Functions

If two functions f and g are continuous on the closed interval $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{1}$$

IN WORDS The integral of a sum equals the sum of the integrals.

Proof Since f and g are continuous on $[a, b]$, then the definite integral of each function exists. Let F and G be an antiderivative of f and g, respectively, on (a, b). Then $F' = f$ and $G' = g$ on (a, b).

Also, since $(F + G)' = F' + G' = f + g$, then $F + G$ is an antiderivative of $f + g$ on (a, b). Now

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= \Big[F(x) + G(x)\Big]_a^b = [F(b) + G(b)] - [F(a) + G(a)]\\ &= [F(b) - F(a)] + [G(b) - G(a)]\\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

■

EXAMPLE 1 Using Property (1) of the Definite Integral

$$\begin{aligned}\int_0^1 (x^2 + e^x)\,dx &= \int_0^1 x^2 dx + \int_0^1 e^x\,dx = \left[\frac{x^3}{3}\right]_0^1 + \Big[e^x\Big]_0^1\\ &= \left[\frac{1^3}{3} - 0\right] + \left[e^1 - e^0\right] = \frac{1}{3} + e - 1 = e - \frac{2}{3}\end{aligned}$$

■

NOW WORK Problem 13.

THEOREM The Integral of a Constant Times a Function

Suppose a function f is continuous on the closed interval $[a, b]$. If k is a constant, then

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \tag{2}$$

IN WORDS A constant factor can be factored out of an integral.

You are asked to prove this theorem in Problem 91.

EXAMPLE 2 **Using Property (2) of the Definite Integral**

$$\int_1^e \frac{3}{x}\,dx = 3\int_1^e \frac{1}{x}dx = 3\Big[\ln|x|\Big]_1^e = 3(\ln e - \ln 1) = 3(1-0) = 3$$ ■

NOW WORK **Problem 15.**

The two properties above can be extended as follows:

THEOREM

Suppose each of the functions $f_1, f_2, \ldots, f_n$ is continuous on the closed interval $[a, b]$. If $k_1, k_2, \ldots, k_n$ are constants, then

$$\begin{aligned}&\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\,dx \\ &\quad = k_1\int_a^b f_1(x)\,dx + k_2\int_a^b f_2(x)\,dx + \cdots + k_n\int_a^b f_n(x)\,dx\end{aligned} \tag{3}$$

You are asked to prove this theorem in Problem 94.

EXAMPLE 3 **Using Property (3) of the Definite Integral**

Find $\displaystyle\int_1^2 \frac{3x^3 - 6x^2 - 5x + 4}{2x}dx$.

Solution The function $f(x) = \dfrac{3x^3 - 6x^2 - 5x + 4}{2x}$ is continuous on the closed interval $[1, 2]$. Using algebra and the properties of the definite integral, we get

$$\begin{aligned}\int_1^2 \frac{3x^3 - 6x^2 - 5x + 4}{2x}dx &= \int_1^2 \left[\frac{3}{2}x^2 - 3x - \frac{5}{2} + \frac{2}{x}\right]dx \quad \text{Simplify} \\ &= \int_1^2 \frac{3}{2}x^2 dx - \int_1^2 3x\,dx - \int_1^2 \frac{5}{2}\,dx + \int_1^2 \frac{2}{x}\,dx \\ &= \frac{3}{2}\int_1^2 x^2\,dx - 3\int_1^2 x\,dx - \frac{5}{2}\int_1^2 dx + 2\int_1^2 \frac{1}{x}\,dx \\ &= \frac{3}{2}\left[\frac{x^3}{3}\right]_1^2 - 3\left[\frac{x^2}{2}\right]_1^2 - \frac{5}{2}\Big[x\Big]_1^2 + 2\Big[\ln|x|\Big]_1^2 \\ &= \frac{1}{2}(8-1) - 3\left(2 - \frac{1}{2}\right) - \frac{5}{2}(2-1) + 2(\ln 2 - \ln 1) \\ &= -\frac{7}{2} + 2\ln 2 \approx -2.114\end{aligned}$$ ■

NOW WORK **Problem 27.**

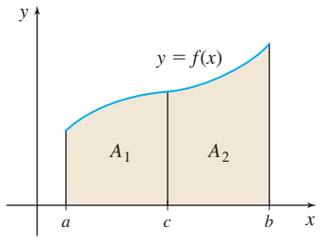
The next property states that a definite integral of a function f from a to b can be evaluated in pieces.

THEOREM

If a function f is continuous on an interval containing the numbers a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{4}$$

A proof of this theorem is given in Appendix B.



A = Area under the graph of f from a to b
= area A_1 + area A_2
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Figure 24

In particular, if f is continuous and nonnegative on a closed interval $[a, b]$ and if c is a number between a and b, then this property has a simple geometric interpretation, as seen in Figure 24.

EXAMPLE 4 Using Property (4) of the Definite Integral

(a) If f is continuous on the closed interval $[2, 7]$, then

$$\int_2^7 f(x)\,dx = \int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$$

(b) If g is continuous on the closed interval $[3, 25]$, then

$$\int_3^{10} g(x)\,dx = \int_3^{25} g(x)\,dx + \int_{25}^{10} g(x)\,dx$$

■

Example 4(b) illustrates that the number c need not lie between a and b.

NOW WORK Problem 43.

The next example illustrates that property (4) is useful when integrating piecewise-defined functions.

EXAMPLE 5 Using Property (4) of the Definite Integral

Find the area A under the graph of

$$f(x) = \begin{cases} x^2 & \text{if} \quad 0 \le x < 10 \\ 100 & \text{if} \quad 10 \le x \le 15 \end{cases}$$

from 0 to 15.

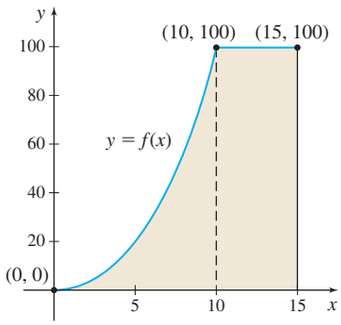


Figure 25 $A = \int_0^{15} f(x)\,dx$

Solution See Figure 25. Since f is nonnegative on the closed interval $[0, 15]$, then $\int_0^{15} f(x)\,dx$ equals the area A under the graph of f from 0 to 15. Since f is continuous on $[0, 15]$,

$$\int_0^{15} f(x)\,dx = \int_0^{10} f(x)\,dx + \int_{10}^{15} f(x)\,dx = \int_0^{10} x^2 dx + \int_{10}^{15} 100\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^{10} + \left[100x\right]_{10}^{15} = \frac{1000}{3} + 500 = \frac{2500}{3}$$

The area under the graph of f is approximately 833.3 square units. ■

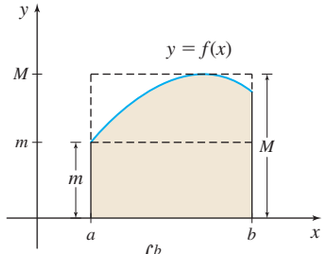
NOW WORK Problem 35.

The next property establishes bounds on a definite integral.

THEOREM Bounds on an Integral

If a function f is continuous on a closed interval $[a, b]$ and if m and M denote the absolute minimum value and the absolute maximum value, respectively, of f on $[a, b]$, then

$$\boxed{m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)} \tag{5}$$



$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

Figure 26

A proof of this theorem is given in Appendix B.

If f is nonnegative on $[a, b]$, then the inequalities in (5) have a geometric interpretation. In Figure 26, the area of the shaded region is $\int_a^b f(x)\,dx$. The smaller rectangle has width $b - a$, height m, and area equal $m(b - a)$. The larger rectangle has width $b - a$, height M, and area $M(b - a)$. These three areas are numerically related by the inequalities in the theorem.

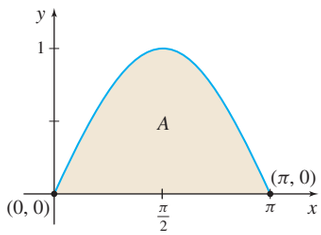


Figure 27 $f(x) = \sin x, 0 \le x \le \pi$.

NEED TO REVIEW? The Extreme Value Theorem is discussed in Section 4.2, pp. xx–xx.

EXAMPLE 6 Using Property (5) of Definite Integrals

(a) Find an upper estimate and a lower estimate for the area A under the graph of $f(x) = \sin x$ from 0 to π.

(b) Find the actual area under the graph.

Solution The graph of f is shown in Figure 27. Since $f(x) \ge 0$ for all x in the closed interval $[0, \pi]$, the area A under its graph is given by the definite integral, $\int_0^\pi \sin x \, dx$.

(a) From the Extreme Value Theorem, f has an absolute minimum value and an absolute maximum value on the interval $[0, \pi]$. The absolute maximum of f occurs at $x = \dfrac{\pi}{2}$, and its value is $f\left(\dfrac{\pi}{2}\right) = \sin\dfrac{\pi}{2} = 1$. The absolute minimum occurs at $x = 0$ and at $x = \pi$; the absolute minimum value is $f(0) = \sin 0 = 0 = f(\pi)$. Using the inequalities in (5), the area under the graph of f is bounded as follows:

$$0 \le \int_0^\pi \sin x \, dx \le \pi$$

(b) The actual area under the graph is

$$A = \int_0^\pi \sin x \, dx = \Big[-\cos x \Big]_0^\pi = -\cos\pi + \cos 0 = 1 + 1 = 2 \text{ square units} \quad \blacksquare$$

NOW WORK Problem 53.

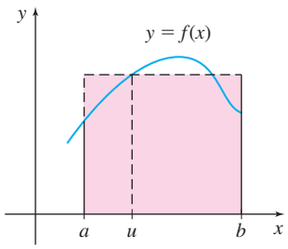


Figure 28 $\int_a^b f(x)\,dx = f(u)(b-a)$.

2 Work with the Mean Value Theorem for Integrals

Suppose f is a function that is continuous and nonnegative on a closed interval $[a, b]$. Figure 28 suggests that the area under the graph of f from a to b, $\int_a^b f(x)\,dx$, is equal to the area of some rectangle of width $b - a$ and height $f(u)$ for a special choice (or choices) of u in the interval $[a, b]$.

In more general terms, for every function f that is continuous on a closed interval $[a, b]$, there is some number u (not necessarily unique) in the interval $[a, b]$ for which

$$\int_a^b f(x)\,dx = f(u)(b-a)$$

This result is known as the *Mean Value Theorem for Integrals.*

THEOREM Mean Value Theorem for Integrals

If a function f is continuous on a closed interval $[a, b]$, there is a real number u, $a \le u \le b$, for which

$$\boxed{\int_a^b f(x)\,dx = f(u)(b-a)} \tag{6}$$

Proof Let f be a function that is continuous on a closed interval $[a, b]$.

If f is a constant function, say, $f(x) = k$, on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b k\,dx = k(b-a) = f(u)(b-a)$$

for any choice of u in $[a, b]$.

If f is not a constant function on $[a, b]$, then by the Extreme Value Theorem, f has an absolute maximum and an absolute minimum on $[a, b]$. Suppose f assumes its absolute minimum at the number c so that $f(c) = m$; and suppose f assumes its absolute maximum at the number C so that $f(C) = M$. Then by the Bounds on an

Integral Theorem (5), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \qquad \text{for all } x \text{ in } [a,b]$$

Divide each part by $(b-a)$ and replace m by $f(c)$ and M by $f(C)$. Then

$$f(c) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(C)$$

Since $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is a real number between $f(c)$ and $f(C)$, it follows from the Intermediate Value Theorem that there is a real number u between c and C, for which

$$f(u) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

That is, there is a real number u, $a \le u \le b$, for which

$$\int_a^b f(x)\,dx = f(u)\,(b-a)$$ ■

NEED TO REVIEW? The Intermediate Value Theorem is discussed in Section 1.3, pp. xx–xx.

EXAMPLE 7 Using the Mean Value Theorem for Integrals

Find the number(s) u guaranteed by the Mean Value Theorem for Integrals for $\int_2^6 x^2 dx$.

Solution The Mean Value Theorem for Integrals states there is a number u, $2 \le u \le 6$, for which

$$\int_2^6 x^2 dx = f(u)(6-2) = 4u^2 \qquad f(u) = u^2$$

Integrate to obtain

$$\int_2^6 x^2 dx = \left[\frac{x^3}{3}\right]_2^6 = \frac{1}{3}(216-8) = \frac{208}{3}$$

Then

$$\frac{208}{3} = 4u^2$$

$$u^2 = \frac{52}{3} \qquad 2 \le u \le 6$$

$$u = \sqrt{\frac{52}{3}} \approx 4.163 \qquad \text{Disregard the negative solution since } u > 0.$$ ■

NOW WORK Problem 55.

3 Find the Average Value of a Function

We know from the Mean Value Theorem for Integrals that if a function f is continuous on a closed interval $[a,b]$, there is a real number u, $a \le u \le b$, for which

$$\int_a^b f(x)\,dx = f(u)(b-a)$$

This means that if the function f is also nonnegative on the closed interval $[a,b]$, the area enclosed by a rectangle of height $f(u)$ and width $b-a$ equals the area under the graph of f from a to b. See Figure 29.

So if we replace $f(x)$ on $[a,b]$ by $f(u)$, we get the same area. Consequently, $f(u)$ can be thought of as an *average value,* or *mean value, of* f *over* $[a,b]$.

We can obtain the average value of f over $[a,b]$ for any function f that is continuous on the closed interval $[a,b]$ by partitioning $[a,b]$ into n subintervals

$$[a,x_1], \quad [x_1,x_2], \quad \ldots, \quad [x_{i-1},x_i], \quad \ldots, \quad [x_{n-1},b]$$

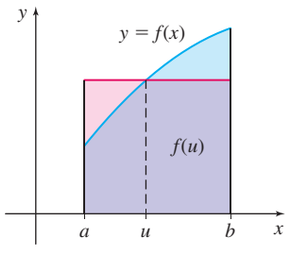


Figure 29 $\displaystyle\int_a^b f(x)\,dx = f(u)(b-a)$

each of length $\Delta x = \dfrac{b-a}{n}$, and choosing a number u_i in each of the n subintervals. Then an approximation of the average value of f over the interval $[a, b]$ is

$$\frac{f(u_1) + f(u_2) + \cdots + f(u_n)}{n} \tag{7}$$

Now we multiply (7) by $\dfrac{b-a}{b-a}$ to obtain

$$\begin{aligned}\frac{f(u_1) + f(u_2) + \cdots + f(u_n)}{n} &= \frac{1}{b-a}\left[f(u_1)\frac{b-a}{n} + f(u_2)\frac{b-a}{n} + \cdots + f(u_n)\frac{b-a}{n}\right] \\ &= \frac{1}{b-a}[f(u_1)\,\Delta x + f(u_2)\,\Delta x + \cdots + f(u_n)\,\Delta x] \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(u_i)\,\Delta x\end{aligned}$$

This sum is an approximation to the average value of f. As the length of each subinterval gets smaller and smaller, the sums become better and better approximations to the average value of f on $[a, b]$. Furthermore, $\sum_{i=1}^{n} f(u_i)\,\Delta x$ are Riemann sums, so $\lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\,\Delta x$ is a definite integral. This suggests the following definition:

DEFINITION Average Value of a Function over an Interval

Let f be a function that is continuous on the closed interval $[a, b]$. The **average value $\bar{y}$ of f over $[a, b]$** is

$$\boxed{\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx} \tag{8}$$

IN WORDS The average value $\bar{y} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ of a function f equals the value $f(u)$ in the Mean Value Theorem for Integrals.

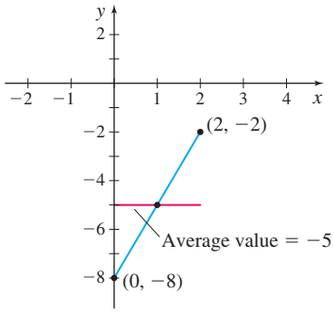


Figure 30 $f(x) = 3x - 8,\ 0 \le x \le 2$

EXAMPLE 8 Finding the Average Value of a Function

Find the average value of $f(x) = 3x - 8$ on the closed interval $[0, 2]$.

Solution The average value of $f(x) = 3x - 8$ on the closed interval $[0, 2]$ is given by

$$\begin{aligned}\bar{y} &= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 (3x-8)dx = \frac{1}{2}\left[\frac{3x^2}{2} - 8x\right]_0^2 \\ &= \frac{1}{2}(6-16) = -5\end{aligned}$$

The average value of f on $[0, 2]$ is $\bar{y} = -5$. ■

The function f and its average value are graphed in Figure 30.

NOW WORK Problem 61.

5.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\displaystyle\int_2^3 (x^2 + x)\,dx = \int_2^3 x^2 dx + \int_2^3 x\,dx$

2. *True or False* $\displaystyle\int_0^3 5e^{x^2} dx = \int_0^3 5dx \cdot \int_0^3 e^{x^2} dx$

3. *True or False* $\displaystyle\int_0^5 (x^3+1)dx = \int_0^{-3} (x^3+1)dx + \int_{-3}^5 (x^3+1)dx$

4. If f is continuous on an interval containing the numbers a, b, and c, and if $\int_a^c f(x)\,dx = 3$ and $\int_c^b f(x)\,dx = -5$, then $\int_a^b f(x)\,dx =$ _____.

5. If a function f is continuous on the closed interval $[a, b]$, then $\bar{y} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is the _____ _____ of f over $[a, b]$.

6. *True or False* If a function f is continuous on a closed interval $[a, b]$ and if m and M denote the absolute minimum value and the absolute maximum value, respectively, of f on $[a, b]$, then

$$m \le \int_a^b f(x)\,dx \le M.$$

Skill Building

In Problems 7–12, find each definite integral given that $\int_1^3 f(x)\,dx = 5$, $\int_1^3 g(x)\,dx = -2$, $\int_3^5 f(x)\,dx = 2$, $\int_3^5 g(x)\,dx = 1$.

7. $\displaystyle\int_1^3 [f(x) - g(x)]\,dx$

8. $\displaystyle\int_1^3 [f(x) + g(x)]\,dx$

9. $\displaystyle\int_1^3 [5f(x) - 3g(x)]\,dx$

10. $\displaystyle\int_1^3 [3f(x) + 4g(x)]\,dx$

11. $\displaystyle\int_1^5 [2f(x) - 3g(x)]\,dx$

12. $\displaystyle\int_1^5 [f(x) - g(x)]\,dx$

In Problems 13–32, find each definite integral using the Fundamental Theorem of Calculus.

13. $\displaystyle\int_0^1 (t^2 - t^{3/2})\,dt$

14. $\displaystyle\int_{-2}^0 (x + x^2)\,dx$

15. $\displaystyle\int_{\pi/2}^{\pi} 4\sin x\,dx$

16. $\displaystyle\int_0^1 3x^2 dx$

17. $\displaystyle\int_1^e -\frac{3}{x}\,dx$

18. $\displaystyle\int_e^8 \frac{1}{2x} dx$

19. $\displaystyle\int_{-\pi/4}^{\pi/4} (1 + 2\sec x\,\tan x)\,dx$

20. $\displaystyle\int_0^{\pi/4} (1 + \sec^2 x)\,dx$

21. $\displaystyle\int_1^4 (\sqrt{x} - 4x)\,dx$

22. $\displaystyle\int_0^1 (\sqrt[5]{t^2} + 1)\,dt$

23. $\displaystyle\int_{-2}^3 [(x-1)(x+3)]\,dx$

24. $\displaystyle\int_0^1 (z^2 + 1)^2 dz$

25. $\displaystyle\int_1^2 \frac{x^2 - 12}{x^4}\,dx$

26. $\displaystyle\int_1^e \frac{5s^2 + s}{s^2} ds$

27. $\displaystyle\int_1^4 \frac{x+1}{\sqrt{x}} dx$

28. $\displaystyle\int_1^9 \frac{\sqrt{x} + 1}{x^2} dx$

29. $\displaystyle\int_1^2 \frac{2x^4 + 1}{x^4} dx$

30. $\displaystyle\int_1^3 \frac{2 - x^2}{x^4} dx$

31. $\displaystyle\int_0^{1/2} \left(5 + \frac{1}{\sqrt{1 - x^2}}\right) dx$

32. $\displaystyle\int_0^1 \left(1 + \frac{5}{1 + x^2}\right) dx$

In Problems 33–38, use properties of integrals and the Fundamental Theorem of Calculus to find each integral.

33. $\displaystyle\int_{-1}^1 f(x)\,dx$, where $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \ge 0 \end{cases}$

34. $\displaystyle\int_{-1}^1 f(x)\,dx$, where $f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \ge 0 \end{cases}$

35. $\displaystyle\int_{-2}^2 f(x)\,dx$, where $f(x) = \begin{cases} 3x & \text{if } -2 \le x < 0 \\ 2x^2 & \text{if } 0 \le x \le 2 \end{cases}$

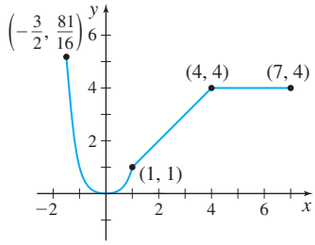
36. $\displaystyle\int_0^4 h(x)\,dx$, where $h(x) = \begin{cases} x - 2 & \text{if } 0 \le x \le 2 \\ 2 - x & \text{if } 2 < x \le 4 \end{cases}$

37. $\displaystyle\int_{-2}^1 H(x)\,dx$, where $H(x) = \begin{cases} 1 + x^2 & \text{if } -2 \le x < 0 \\ 1 + 3x & \text{if } 0 \le x \le 1 \end{cases}$

38. $\displaystyle\int_{-\pi/2}^{\pi/2} f(x)\,dx$, where $f(x) = \begin{cases} x^2 + x & \text{if } -\dfrac{\pi}{2} \le x \le 0 \\ \sin x & \text{if } 0 < x < \dfrac{\pi}{4} \\ \dfrac{\sqrt{2}}{2} & \text{if } \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \end{cases}$

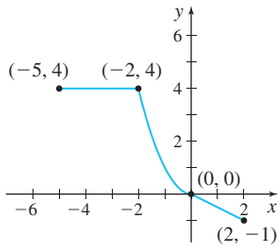
In Problems 39–42 the domain of f is a closed interval $[a, b]$. Find $\int_a^b f(x)\,dx$.

39.



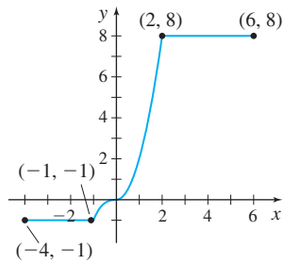
$$f(x) = \begin{cases} x^4 \text{ if } -\frac{3}{2} \le x < 1 \\ x \text{ if } 1 \le x < 4 \\ 4 \text{ if } 4 \le x \le 7 \end{cases}$$

40.



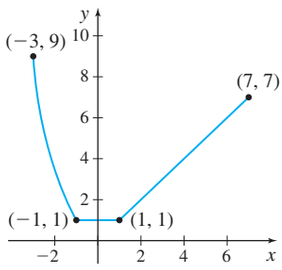
$$f(x) = \begin{cases} 4 \text{ if } -5 \le x < -2 \\ x^2 \text{ if } -2 \le x \le 0 \\ -\frac{x}{2} \text{ if } 0 < x \le 2 \end{cases}$$

41.



$$f(x) = \begin{cases} -1 \text{ if } -4 \le x \le -1 \\ x^3 \text{ if } -1 < x < 2 \\ 8 \text{ if } 2 \le x \le 6 \end{cases}$$

42.



$$f(x) = \begin{cases} x^2 \text{ if } -3 \le x \le -1 \\ 1 \text{ if } -1 < x \le 1 \\ x \text{ if } 1 < x \le 7 \end{cases}$$

In Problems 43–46, use properties of definite integrals to verify each statement. Assume that all integrals involved exist.

43. $\displaystyle\int_3^{11} f(x)\,dx - \int_7^{11} f(x)\,dx = \int_3^7 f(x)\,dx$

44. $\displaystyle\int_{-2}^6 f(x)\,dx - \int_3^6 f(x)\,dx = \int_{-2}^3 f(x)\,dx$

45. $\displaystyle\int_0^4 f(x)\,dx - \int_6^4 f(x)\,dx = \int_0^6 f(x)\,dx$

46. $\displaystyle\int_{-1}^3 f(x)\,dx - \int_5^3 f(x)\,dx = \int_{-1}^5 f(x)\,dx$

In Problems 47–54, use the Bounds on an Integral Theorem to obtain a lower estimate and an upper estimate for each integral.

47. $\int_1^3 (5x+1)\,dx$

48. $\int_0^1 (1-x)\,dx$

49. $\int_{\pi/4}^{\pi/2} \sin x\,dx$

50. $\int_{\pi/6}^{\pi/3} \cos x\,dx$

51. $\int_0^1 \sqrt{1+x^2}\,dx$

52. $\int_{-1}^1 \sqrt{1+x^4}\,dx$

53. $\int_0^1 e^x\,dx$

54. $\int_1^{10} \frac{1}{x}\,dx$

In Problems 55–60, for each integral find the number(s) u guaranteed by the Mean Value Theorem for Integrals.

55. $\int_0^3 (2x^2+1)\,dx$

56. $\int_0^2 (2-x^3)\,dx$

57. $\int_0^4 x^2\,dx$

58. $\int_0^4 (-x)\,dx$

59. $\int_0^{2\pi} \cos x\,dx$

60. $\int_{-\pi/4}^{\pi/4} \sec x \tan x\,dx$

In Problems 61–70, find the average value of each function f over the given interval.

61. $f(x) = e^x$ over $[0, 1]$

62. $f(x) = \frac{1}{x}$ over $[1, e]$

63. $f(x) = x^{2/3}$ over $[-1, 1]$

64. $f(x) = \sqrt{x}$ over $[0, 4]$

65. $f(x) = \sin x$ over $\left[0, \frac{\pi}{2}\right]$

66. $f(x) = \cos x$ over $\left[0, \frac{\pi}{2}\right]$

67. $f(x) = 1 - x^2$ over $[-1, 1]$

68. $f(x) = 16 - x^2$ over $[-4, 4]$

69. $f(x) = e^x - \sin x$ over $\left[0, \frac{\pi}{2}\right]$

70. $f(x) = x + \cos x$ over $\left[0, \frac{\pi}{2}\right]$

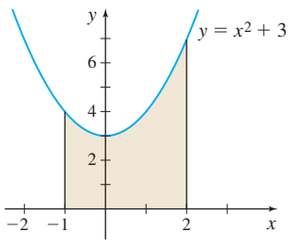
In Problems 71–74, find:

(a) *The area under the graph of the function over the indicated interval.*

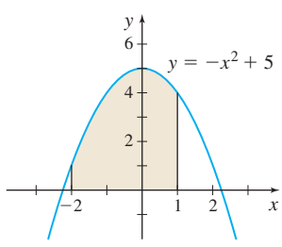
(b) *The average value of each function over the indicated interval.*

(c) *Interpret the results geometrically.*

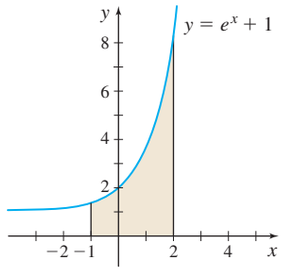
71.



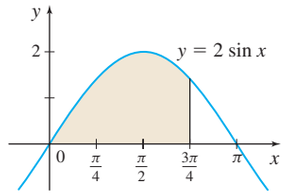
72.



73.



74.



Applications and Extensions

In Problems 75–78, find each definite integral using the Fundamental Theorem of Calculus and properties of definite integrals.

75. $\int_{-2}^3 (x + |x|)\,dx$

76. $\int_0^3 |x-1|\,dx$

77. $\int_0^2 |3x-1|\,dx$

78. $\int_0^2 |2-x|\,dx$

79. Average Temperature A rod 3 meters (m) long is heated to $25x$ °C, where x is the distance in meters from one end of the rod. Find the average temperature of the rod.

80. Average Daily Rainfall The rainfall per day, x days after the beginning of the year, is modeled by the function $r(x) = 0.00002(6511 + 366x - x^2)$, measured in centimeters. Find the average daily rainfall for the first 180 days of the year.

81. Structural Engineering A structural engineer designing a member of a structure must consider the forces that will act on that member. Most often, natural forces like snow, wind, or rain distribute force over the entire member. For practical purposes, however, an engineer determines the distributed force as a single resultant force acting at one point on the member. If the distributed force is given by the function $W = W(x)$, in Newtons per meter (N/m), then the magnitude F_R of the resultant force is

$$F_R = \int_a^b W(x)\,dx$$

The position $\bar{x}$ of the resultant force measured in meters from the origin is given by

$$\overline{x} = \frac{\int_a^b x\,W(x)\,dx}{\int_a^b W(x)\,dx}$$

If the distributed force is $W(x) = 0.75x^3$, $0 \le x \le 5$, find:

(a) The magnitude of the resultant force.

(b) The position from the origin of the resultant force.

Source: Problem contributed by the students at Trine University, Avalon, IN.

82. Chemistry: Enthalpy In chemistry, **enthalpy** is a measure of the total energy of a system. For a nonreactive process with no phase change, the change in enthalpy ΔH is given by $\Delta H = \int_{T_1}^{T_2} C_p\,dT$, where C_p is the specific heat of the system in question. The specific heat of the chemical benzene is

$$C_p = 0.126 + (2.34 \times 10^{-6})T,$$

where C_p is in kJ/(mol °C), and T is in degrees Celsius.

(a) What are the units of the change in enthalpy ΔH?

(b) What is the change in enthalpy ΔH associated with increasing the temperature of benzene from 20 °C to 40 °C?

(c) What is the change in enthalpy ΔH associated with increasing the temperature of benzene from 20 °C to 60 °C?

(d) What is happening to the enthalpy as the temperature increases?

Source: Problem contributed by the students at Trine University, Avalon, IN.

83. Average Mass Density The mass density of a metal bar of length 3 meters (m) is given by $\rho(x) = 1000 + x - \sqrt{x}$ kilograms per cubic meter (kg/m^3), where x is the distance in meters from one end of the bar. What is the average mass density over the length of the entire bar?

84. Average Velocity The acceleration at time t of an object in rectilinear motion is given by $a(t) = 4\pi \cos t$. If the object's velocity is 0 at $t = 0$, what is the average velocity of the object over the interval $0 \le t \le \pi$?

85. Average Area What is the average area of all circles whose radii are between 1 and 3 m?

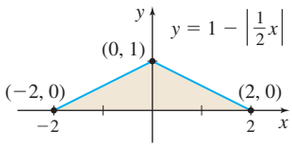
86. Area

(a) Use properties of integrals and the Fundamental Theorem of Calculus to find the area under the graph of $y = 3 - |x|$ from -3 to 3.

(b) Check your answer by using elementary geometry.

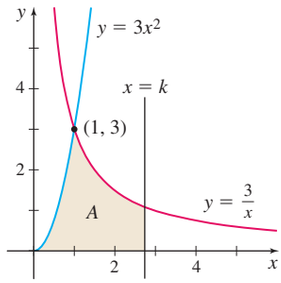
87. Area

(a) Use properties of integrals and the Fundamental Theorem of Calculus to find the area under the graph of $y = 1 - \left|\frac{1}{2}x\right|$ from -2 to 2.



(b) Check your answer by using elementary geometry. See the figure.

88. Area Let A be the area in the first quadrant that is enclosed by the graphs of $y = 3x^2$, $y = \frac{3}{x}$, the x-axis, and the line $x = k$, where $k > 1$, as shown in the figure.



(a) Find the area A as a function of k.

(b) When the area is 7, what is k?

(c) If the area A is increasing at the constant rate of 5 square units per second, at what rate is k increasing when $k = 15$?

89. Rectilinear Motion A car starting from rest accelerates at the rate of 3 meters per second squared (m/s^2). Find its average speed over the first 8 s.

90. Rectilinear Motion A car moving at a constant velocity of 80 miles per hour (mph) begins to decelerate at the rate of 10 mi/h^2. Find its average speed over the next 10 minutes.

91. Average Slope

(a) Use the definition of average value of a function to find the *average slope* of the graph of $y = f(x)$, where $a \le x \le b$. (Assume that f' is continuous.)

(b) Give a geometric interpretation.

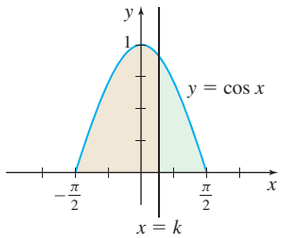
92. What theorem guarantees that the average slope found in Problem 91 is equal to $f'(u)$ for some u in $[a, b]$? What *different* theorem guarantees the same thing? (The connection between these theorems should now become apparent.)

93. Prove that if a function f is continuous on a closed interval $[a, b]$ and if k is a constant, then $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.

94. Prove that if the functions $f_1, f_2, \ldots, f_n$ are continuous on a closed interval $[a, b]$ and if $k_1, k_2, \ldots, k_n$ are constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\,dx$$
$$= k_1\int_a^b f_1(x)\,dx + k_2\int_a^b f_2(x)\,dx + \cdots + k_n\int_a^b f_n(x)\,dx$$

95. Area The area under the graph of $y = \cos x$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ is separated into two parts by the line $x = k$, $\frac{-\pi}{2} < k < \frac{\pi}{2}$, as shown in the figure. If the area under the graph of y from $-\frac{\pi}{2}$ to k is three times the area under the graph of y from k to $\frac{\pi}{2}$, find k.



96. Displacement of a Damped Spring The displacement x in meters (m) of a damped spring from its equilibrium position at time t seconds is given by

$$x(t) = \frac{\sqrt{15}}{10}e^{-t}\sin\left(\sqrt{15}t\right) + \frac{3}{2}e^{-t}\cos\left(\sqrt{15}t\right)$$

(a) What is the displacement of the spring at $t = 0$?

(b) Graph the displacement for the first 2 seconds of the springs' motion.

CAS **(c)** Find the average displacement of the spring for the first 2 seconds of its motion.

CAS **97. Area** Let

$$f(x) = |x^4 + 3.44x^3 - 0.5041x^2 - 5.0882x + 1.1523|$$

be defined on the interval $[-3, 1]$. Find the area under the graph of f.

98. If f is continuous on $[a, b]$, show that the functions defined by

$$F(x) = \int_c^x f(t)\,dt \qquad G(x) = \int_d^x f(t)\,dt$$

for any choice of c and d in (a, b) always differ by a constant. Also show that

$$F(x) - G(x) = \int_c^d f(t)\,dt$$

99. Put It Together Suppose $a < c < b$ and the function f is continuous on $[a, b]$ and differentiable on (a, b). Which of the following is *not* necessarily true?

(a) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

(b) There is a number d in $[a, b]$ for which

$$f'(d) = \frac{f(b) - f(a)}{b - a}$$

(c) $\displaystyle\int_a^b f(x)\,dx \geq 0.$

(d) $\displaystyle\lim_{x \to c} f(x) = f(c).$

(e) If k is a real number, then $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$

100. Minimizing Area Find $b > 0$ so that the area enclosed in the first quadrant by the graph of $y = 1 + b - bx^2$ and the coordinate axes is a minimum.

101. Area Find the area enclosed by the graph of $\sqrt{x} + \sqrt{y} = 1$ and the coordinate axes.

Challenge Problems

102. Average Velocity For a freely falling object starting from rest, $v_0 = 0$, find:

(a) The average velocity $\overline{v}_t$ with respect to the time t in seconds over the closed interval $[0, 5]$.

(b) The average velocity $\overline{v}_s$ with respect to the distance s of the object from its position at $t = 0$ over the closed interval $[0, s_1]$, where s_1 is the distance the object falls from $t = 0$ to $t = 5$ seconds.

(*Hint:* The derivation of the formulas for freely falling objects is given in Section 4.8, pp. xx–xx.)

103. Average Velocity If an object falls from rest for 3 s, find:

(a) Its average velocity with respect to time.

(b) Its average velocity with respect to the distance it travels in 3 seconds.

104. Free Fall For a freely falling object starting from rest, $v_0 = 0$, find:

(a) The average velocity $\overline{v}_t$ with respect to the time t over the closed interval $[0, t_1]$.

(b) The average velocity $\overline{v}_s$ with respect to the distance s of the object from its position at $t = 0$ over the closed interval $[0, s_1]$, where s_1 is the distance the object falls in time t_1. Assume $s(0) = 0$.

105. Put It Together

(a) What is the domain of $f(x) = 2|x - 1|x^2$?

(b) What is the range of f?

(c) For what values of x is f continuous?

(d) For what values of x is the derivative of f continuous?

(e) Find $\int_0^1 f(x)\,dx$.

106. Probability A function f that is continuous on the closed interval $[a, b]$, and for which (i) $f(x) \geq 0$ for numbers x in $[a, b]$ and (ii) $\int_a^b f(x)\,dx = 1$, is called a **probability density function** on $[a, b]$. If $a \leq c < d \leq b$, the probability of obtaining a value between c and d is defined as $\int_c^d f(x)\,dx$.

(a) Find a constant k so that $f(x) = kx$ is a probability density function on $[0, 2]$.

(b) Find the probability of obtaining a value between 1 and 1.5.

107. Cumulative Probability Distribution Refer to Problem 106. If f is a probability density function on an interval $[a, b]$, the **cumulative distribution function** F for f is defined as

$$F(x) = \int_a^x f(t)\,dt, \qquad a \leq x \leq b$$

Find the cumulative distribution function F for the probability density function $f(x) = kx$ of Problem 106(a).

108. For the cumulative distribution function $F(x) = x - 1$, on the interval $[1, 2]$:

(a) Find the probability density function f corresponding to F.

(b) Find the probability of obtaining a value between 1.5 and 1.7.

109. Let $f(x) = x^3 - 6x^2 + 11x - 6$. Find $\int_1^3 |f(x)|\,dx$.

110. Show that for $x > 1$, $\ln x < 2(\sqrt{x} - 1)$.

(*Hint*: Use the result from Problem 114.)

111. Prove that the average value of a line segment $y = m(x - x_1) + y_1$ on the interval $[x_1, x_2]$ equals the y-coordinate of the midpoint of the line segment from x_1 to x_2.

112. Prove that if a function f is continuous on a closed interval $[a, b]$ and if $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

113. (a) Prove that if f is continuous on a closed interval $[a, b]$ and $\int_a^b f(x)\,dx = 0$, there is at least one number c in $[a, b]$ for which $f(c) = 0$.

(b) Give a counterexample to the statement above if f is not required to be continuous.

114. Prove that if functions f and g are continuous on a closed interval $[a, b]$ and if $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

115. Prove that if f is continuous on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Give a geometric interpretation of the inequality.

5.5 The Indefinite Integral; Growth and Decay Models

OBJECTIVES *When you finish this section, you should be able to:*

1 Find indefinite integrals (p. 37)
2 Use properties of indefinite integrals (p. 38)
3 Solve differential equations involving growth and decay (p. 40)

The Fundamental Theorem of Calculus establishes an important relationship between definite integrals and antiderivatives: the definite integral $\int_a^b f(x)\,dx$ can be found easily if an antiderivative of f can be found. Because of this, it is customary to use the integral symbol $\int$ as an instruction to find all antiderivatives of a function.

DEFINITION Indefinite Integral

The expression $\int f(x)\,dx$, called the **indefinite integral of f**, is defined as,

$$\boxed{\int f(x)\,dx = F(x) + C}$$

where F is any function for which $\dfrac{d}{dx}F(x) = f(x)$ and C is a number, and called the **constant of integration**.

CAUTION In writing an indefinite integral $\int f(x)\,dx$, remember to include the "dx."

For example,

$$\int (x^2 + 1)\,dx = \frac{x^3}{3} + x + C \qquad \frac{d}{dx}\left(\frac{x^3}{3} + x + C\right) = \frac{3x^2}{3} + 1 + 0 = x^2 + 1$$

The process of finding either the indefinite integral $\int f(x)\,dx$ or the definite integral $\int_a^b f(x)\,dx$ is called **integration**, and in both cases the function f is called the **integrand**.

NOTE The definite integral $\int_a^b f(x)\,dx$ is a number; the indefinite integral $\int f(x)\,dx$ is a family of functions.

It is important to distinguish between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$. The definite integral is a *number* that depends on the limits of integration a and b. In contrast, the indefinite integral of f is a *family* of functions $F(x) + C$, C a constant, for which $F'(x) = f(x)$. For example,

$$\int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} \qquad \int x^2 dx = \frac{x^3}{3} + C$$

NEED TO REVIEW? Refer to Table 7 in Section 4.8, pp. xx–xx.

We summarize the antiderivatives of some important functions in Table 1 on page 38. Each entry is a result of a differentiation formula.

1 Find Indefinite Integrals

EXAMPLE 1 Finding Indefinite Integrals

Find:

(a) $\int x^4\,dx$ **(b)** $\int \sqrt{x}\,dx$ **(c)** $\int \dfrac{\sin x}{\cos^2 x}\,dx$

Solution **(a)** All the antiderivatives of $f(x) = x^4$ are $F(x) = \dfrac{x^5}{5} + C$, so

$$\int x^4\,dx = \frac{x^5}{5} + C$$

TABLE 1

Table of Integrals

$\int dx = x + C$	$\int \sec x \tan x \, dx = \sec x + C$
$\int x^a \, dx = \frac{x^{a+1}}{a+1} + C; \quad a \neq -1$	$\int \csc x \cot x \, dx = -\csc x + C$
$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln \lvert x \rvert + C$	$\int \csc^2 x \, dx = -\cot x + C$
$\int e^x \, dx = e^x + C$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C, \quad \lvert x \rvert < 1$
$\int a^x \, dx = \frac{a^x}{\ln a} + C; \quad a > 0, a \neq 1$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
$\int \sin x \, dx = -\cos x + C$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C, \quad \lvert x \rvert > 1$
$\int \cos x \, dx = \sin x + C$	$\int \sinh x \, dx = \cosh x + C$
$\int \sec^2 x \, dx = \tan x + C$	$\int \cosh x \, dx = \sinh x + C$

(b) All the antiderivatives of $f(x) = \sqrt{x} = x^{1/2}$ are $F(x) = \frac{x^{3/2}}{\frac{3}{2}} + C = \frac{2x^{3/2}}{3} + C$, so

$$\int \sqrt{x} \, dx = \frac{2x^{3/2}}{3} + C$$

(c) No antiderivative in Table 1 corresponds to $f(x) = \frac{\sin x}{\cos^2 x}$, so we begin by using trigonometric identities to rewrite $\frac{\sin x}{\cos^2 x}$ in a form whose antiderivative is recognizable.

NEED TO REVIEW? Trigonometric identities are discussed in Appendix A.4, pp. A-32 to A-35.

$$\frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x \cdot \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

Then

$$\int \frac{\sin x}{\cos^2 x} \, dx = \int \sec x \tan x \, dx = \sec x + C$$ ■

NOW WORK Problem 5 and 7.

2 Use Properties of Indefinite Integrals

Since the definite integral and the indefinite integral are closely related, properties of indefinite integrals are very similar to those of definite integrals:

- **Derivative of an Integral**:

$$\boxed{\frac{d}{dx}\left[\int f(x) \, dx\right] = f(x)} \tag{1}$$

Property (1) is a consequence of the definition of $\int f(x)\,dx$. For example,

$$\frac{d}{dx}\int \sqrt{x^2+1}\,dx = \sqrt{x^2+1} \qquad \frac{d}{dt}\int e^t \cos t\,dt = e^t \cos t$$

- **Integral of the Sum of Two Functions:**

$$\boxed{\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx} \tag{2}$$

IN WORDS The indefinite integral of a sum of two functions equals the sum of the indefinite integrals.

The proof of property (2) follows directly from properties of derivatives, and is left as an exercise. See Problem 68.

- **Integral of a Constant Times a Function:** If k is a constant,

$$\boxed{\int k\,f(x)\,dx = k\int f(x)\,dx} \tag{3}$$

IN WORDS To find the indefinite integral of a constant k times a function f, find the indefinite integral of f and then multiply by k.

To prove property (3), differentiate the right side of the (3).

$$\frac{d}{dx}\left[k\int f(x)\,dx\right] = k\left[\frac{d}{dx}\int f(x)\,dx\right] = k\,f(x)$$

Constant Multiple Rule; Property (1)

EXAMPLE 2 Using Properties of the Indefinite Integral

$$\int (2x^{1/3}+5x^{-1})\,dx = \int 2x^{1/3}\,dx + \int \frac{5}{x}\,dx = 2\int x^{1/3}\,dx + 5\int \frac{1}{x}\,dx$$
$$= 2\cdot\frac{x^{4/3}}{\frac{4}{3}} + 5\ln|x| + C = \frac{3x^{4/3}}{2} + 5\ln|x| + C$$

NOW WORK Problem 13.

Sometimes an appropriate algebraic manipulation is required before integrating.

EXAMPLE 3 Using Properties of the Indefinite Integral

(a) $$\int \left(\frac{12}{x^5}+\frac{1}{\sqrt{x}}\right)dx = 12\int \frac{1}{x^5}\,dx + \int \frac{1}{\sqrt{x}}\,dx = 12\int x^{-5}\,dx + \int x^{-1/2}\,dx$$
$$= 12\left(\frac{x^{-4}}{-4}\right) + \frac{x^{1/2}}{\frac{1}{2}} + C = -\frac{3}{x^4} + 2\sqrt{x} + C$$

(b) $$\int \frac{x^2+6}{x^2+1}\,dx = \int \frac{(x^2+1)+5}{x^2+1}\,dx = \int \left[\frac{x^2+1}{x^2+1}+\frac{5}{x^2+1}\right]dx$$

Algebra

$$= \int \left[1+\frac{5}{x^2+1}\right]dx = \int dx + \int \frac{5}{x^2+1}\,dx$$

Sum Property

$$= \int dx + 5\int \frac{1}{x^2+1} = x + 5\tan^{-1}x + C$$

NOW WORK Problem 33.

3 Solve Differential Equations Involving Growth and Decay

There are situations in science and nature, such as radioactive decay, population growth, and interest paid on an investment, in which a quantity A varies with time t in such a way that the rate of change of A with respect to t is proportional to A itself. These situations can be modeled by the differential equation

$$\boxed{\frac{dA}{dt} = kA} \tag{4}$$

where $k \neq 0$ is a real number:

- If $k > 0$, then $\frac{dA}{dt} = kA$, the rate of change of A with respect to t is positive, and the amount A is increasing.
- If $k < 0$, then $\frac{dA}{dt} = kA$, the rate of change of A with respect to t is negative, and the amount A is decreasing.

Suppose that the initial amount A_0 of the substance is known, giving us the boundary condition, or initial condition, $A = A(0) = A_0$ when $t = 0$.

We solve differential equations of the form $\frac{dA}{dt} = kA$ by writing $\frac{dA}{dt} = kA$ as $\frac{dA}{A} = k\,dt$.* Then we integrate both sides of the equation, on the left with respect to A and on the right with respect to t.

$$\int \frac{1}{A}\,dA = \int k\,dt$$

$$\ln|A| = kt + C$$

$$\ln A = kt + C \qquad A > 0$$

The initial condition requires that $A = A_0$ when $t = 0$. Then $\ln A_0 = C$, so

$$\ln A = kt + \ln A_0$$

$$\ln A - \ln A_0 = kt$$

$$\ln \frac{A}{A_0} = kt$$

$$\frac{A}{A_0} = e^{kt}$$

$$A = A_0 e^{kt}$$

The solution to the differential equation $\frac{dA}{dt} = kA$ is

$$\boxed{A = A_0 e^{kt}} \tag{5}$$

where A_0 is the initial amount.

Functions $A = A(t)$ whose rates of change are $\frac{dA}{dt} = kA$ are said to follow the **exponential law,** or the **law of uninhibited growth or decay**—or in a business context, **the law of continuously compounded interest.** Figure 31 on page 41, shows the graphs of the function $A(t) = A_0 e^{kt}$ for both $k > 0$ and $k < 0$.

*This technique for solving a differential equation, called **separating the variables**, is discussed in more detail in Chapter 16.

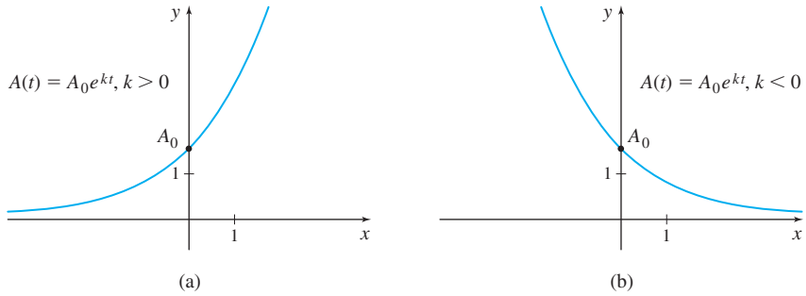


Figure 31

EXAMPLE 4 Solving a Differential Equation for Growth

NOTE Example 4 is a model of uninhibited growth; it accurately reflects growth in early stages. After a time, growth no longer continues at a rate proportional to the number present. Factors, such as disease, lack of space, and dwindling food supply, begin to affect the rate of growth.

Assume that a colony of bacteria grows at a rate proportional to the number of bacteria present. If the number of bacteria doubles in 5 hours (h), how long will it take for the number of bacteria to triple?

Solution Let $N(t)$ be the number of bacteria present at time t. Then the assumption that this colony of bacteria grows at a rate proportional to the number present can be modeled by

$$\frac{dN}{dt} = kN$$

where k is a positive constant of proportionality. To find k, we write the differential equation as $\frac{dN}{N} = k\,dt$ and integrate both sides. This differential equation is of form (4), and its solution is given by (5). So, we have

$$N(t) = N_0 e^{kt}$$

where N_0 is the initial number of bacteria in the colony. Since the number of bacteria doubles to $2N_0$ in 5 h,

$$\begin{aligned} N(5) = N_0\, e^{5k} &= 2N_0 \\ e^{5k} &= 2 \\ k &= \frac{1}{5}\ln 2 \end{aligned}$$

The time t required for this colony to triple obeys the equation

$$\begin{aligned} N(t) &= 3N_0 \\ N_0 e^{kt} &= 3N_0 \\ e^{kt} &= 3 \\ t = \frac{1}{k}\ln 3 &= 5\frac{\ln 3}{\ln 2} \approx 8 \\ \uparrow\ k &= \frac{1}{5}\ln 2 \end{aligned}$$

The number of bacteria will triple in about 8 h. ■

NOW WORK Problem 57.

For a radioactive substance, the **rate of decay** is proportional to the amount of substance present at a given time t. That is, if $A = A(t)$ represents the amount of a radioactive substance at time t, then

$$\boxed{\frac{dA}{dt} = kA}$$

ORIGINS Willard F. Libby (1908–1980) grew up in California and went to college and graduate school at UC Berkeley. Libby was a physical chemist who taught at the University of Chicago and later at UCLA. While at Chicago, he developed the methods for using natural carbon-14 to date archaeological artifacts. Libby won the Nobel Prize in Chemistry in 1960 for this work.

where $k < 0$ and depends on the radioactive substance. The **half-life** of a radioactive substance is the time required for half of the substance to decay.

Carbon dating, a method for determining the age of an artifact, uses the fact that all living organisms contain two kinds of carbon: carbon-12 (a stable carbon) and a small proportion of carbon-14 (a radioactive isotope). When an organism dies, the amount of carbon-12 present remains unchanged, while the amount of carbon-14 begins to decrease. This change in the amount of carbon-14 present relative to the amount of carbon-12 present makes it possible to calculate how long ago the organism died.

EXAMPLE 5 Solving a Differential Equation for Decay

The skull of an animal found in an archaeological dig contains about 20% of the original amount of carbon-14. If the half-life of carbon-14 is 5730 years, how long ago did the animal die?

Solution Let $A = A(t)$ be the amount of carbon-14 present in the skull at time t. Then A satisfies the differential equation $\dfrac{dA}{dt} = kA$, whose solution is

$$A = A_0 e^{kt}$$

where A_0 is the amount of carbon-14 present at time $t = 0$. To determine the constant k, we use the fact that when $t = 5730$, half of the original amount A_0 remains.

$$\frac{1}{2}A_0 = A_0 e^{5730k}$$

$$\frac{1}{2} = e^{5730k}$$

$$5730k = \ln\frac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{5730}$$

The relationship between the amount A of carbon-14 and the time t is

$$A(t) = A_0 e^{(-\ln 2/5730)t}$$

In this skull, 20% of the original amount of carbon-14 remains, so $A(t) = 0.20A_0$.

$$0.20A_0 = A_0 e^{(-\ln 2/5730)t}$$

$$0.20 = e^{(-\ln 2/5730)t}$$

Now, we take the natural logarithm of both sides.

$$\ln 0.20 = -\frac{\ln 2}{5730} \cdot t$$

$$t = -5730 \cdot \frac{\ln 0.20}{\ln 2} \approx 13{,}300$$

The animal died approximately 13,300 years ago. ■

NOW WORK Problem 59.

5.5 Assess Your Understanding

Concepts and Vocabulary

1. $\dfrac{d}{dx}\left[\displaystyle\int f(x)\,dx\right] =$ _____

2. *True or False* If k is a constant, then

$$\int kf(x)\,dx = \left[\int k\,dx\right]\left[\int f(x)\,dx\right]$$

3. If a is a number, then $\int x^a\,dx =$ _____, provided $a \neq -1$.

4. *True or False* When integrating a function f, a constant of integration C is added to the result because $\int f(x)\,dx$ denotes all the antiderivatives of f.

Skill Building

In Problems 5–38, find each indefinite integral.

5. $\int x^{2/3}\,dx$

6. $\int t^{-4}\,dt$

7. $\int \frac{1}{\sqrt{1-x^2}}\,dx$

8. $\int \frac{1}{1+x^2}\,dx$

9. $\int \frac{5x^2+2x-1}{x}\,dx$

10. $\int \frac{x+1}{x}\,dx$

11. $\int \frac{4}{3t}\,dt$

12. $\int 2e^u\,du$

13. $\int (4x^3-3x^2+5x-2)\,dx$

14. $\int (3x^5-2x^4-x^2-1)\,dx$

15. $\int \left(\frac{1}{x^3}+1\right)dx$

16. $\int \left(x-\frac{1}{x^2}\right)dx$

17. $\int (3\sqrt{z}+z)\,dz$

18. $\int (4\sqrt{x}+1)\,dx$

19. $\int (4t^{3/2}+t^{1/2})\,dt$

20. $\int \left(3x^{2/3}-\frac{1}{\sqrt{x}}\right)dx$

21. $\int u(u-1)\,du$

22. $\int t^2(t+1)\,dt$

23. $\int \frac{3x^5+1}{x^2}\,dx$

24. $\int \frac{x^2+2x+1}{x^4}\,dx$

25. $\int \frac{t^2-4}{t-2}\,dt$

26. $\int \frac{z^2-16}{z+4}dz$

27. $\int (2x+1)^2\,dx$

28. $\int 3(x^2+1)^2\,dx$

29. $\int (x+e^x)\,dx$

30. $\int (2e^x-x^3)\,dx$

31. $\int 8(1+x^2)^{-1}dx$

32. $\int \frac{-7}{1+x^2}\,dx$

33. $\int \frac{x^2-1}{2x^3}\,dx$

34. $\int \frac{x^2+4x-1}{x^2}\,dx$

35. $\int \frac{\tan x}{\cos x}\,dx$

36. $\int \frac{1}{\sin^2 x}\,dx$

37. $\int \frac{2}{5\sqrt{1-x^2}}\,dx$

38. $\int -\frac{4}{x\sqrt{x^2-1}}\,dx$

In Problems 39–50, solve each differential equation using the given boundary condition.

39. $\frac{dy}{dx} = e^x$, $y = 4$ when $x = 0$

40. $\frac{dy}{dx} = \frac{1}{x}$, $y = 0$ when $x = 1$

41. $\frac{dy}{dx} = \frac{x^2+x+1}{x}$, $y = 0$ when $x = 1$

42. $\frac{dy}{dx} = x + e^x$, $y = 4$ when $x = 0$

43. $\frac{dy}{dx} = xy^{1/2}$, $y = 1$ when $x = 2$

44. $\frac{dy}{dx} = x^{1/2}y$, $y = 1$ when $x = 0$

45. $\frac{dy}{dx} = \frac{y-1}{x-1}$, $y = 2$ when $x = 2$

46. $\frac{dy}{dx} = \frac{y}{x}$, $y = 2$ when $x = 1$

47. $\frac{dy}{dx} = \frac{x}{\cos y}$, $y = \pi$ when $x = 2$

48. $\frac{dy}{dx} = y\sin x$, $y = e$ when $x = 0$

49. $\frac{dy}{dx} = \frac{4e^x}{y}$, $y = 2$ when $x = 0$

50. $\frac{dy}{dx} = 5ye^x$, $y = 1$ when $x = 0$

Applications and Extensions

51. Uninhibited Growth The population of a colony of mosquitoes obeys the uninhibited growth equation $\frac{dN}{dt} = kN$. If there are 1500 mosquitoes initially, and there are 2500 mosquitoes after 24 h, what is the mosquito population after 3 days?

52. Radioactive Decay A radioactive substance follows the decay equation $\frac{dA}{dt} = kA$. If 25% of the substance disappears in 10 years, what is its half-life?

53. Population Growth The population of a suburb grows at a rate proportional to the population. Suppose the population doubles in size from 4000 to 8000 in an 18-month period and continues at the current rate of growth.

(a) Write a differential equation that models the population P at time t in months.

(b) Find the general solution to the differential equation.

(c) Find the particular solution to the differential equation with the initial condition $P(0) = 4000$.

(d) What will the population be in 4 years [$t = 48$]?

54. Uninhibited Growth At any time t in hours, the rate of increase in the area, in millimeters squared (mm^2), of a culture of bacteria is twice the area A of the culture.

(a) Write a differential equation that models the area of the culture at time t.

(b) Find the general solution to the differential equation.

(c) Find the particular solution to the differential equation if $A = 10\,\text{mm}^2$, when $t = 0$.

55. Radioactive Decay The amount A of the radioactive element radium in a sample decays at a rate proportional to the amount of radium present. The half-life of radium is 1690 years.

(a) Write a differential equation that models the amount A of radium present at time t.

(b) Find the general solution to the differential equation.

(c) Find the particular solution to the differential equation with the initial condition $A(0) = 8\,\text{g}$.

(d) How much radium will be present in the sample in 100 years?

56. Radioactive Decay Carbon-14 is a radioactive element present in living organisms. After an organism dies, the amount A of carbon-14 present begins to decline at a rate proportional to the amount present at the time of death. The half-life of carbon-14 is 5730 years.

(a) Write a differential equation that models the rate of decay of carbon-14.

(b) Find the general solution to the differential equation.

(c) A piece of fossilized charcoal is found that contains 30% of the carbon-14 that was present when the tree it came from died. How long ago did the tree die?

57. World Population Growth Barring disasters (human-made or natural), the population P of humans grows at a rate proportional to its current size. According to the U.N. World Population studies, from 2005 to 2010 the population of the more developed regions of the world (Europe, North America, Australia, New Zealand, and Japan) grew at an annual rate of 0.409% per year.

(a) Write a differential equation that models the growth rate of the population.

(b) Find the general solution to the differential equation.

(c) Find the particular solution to the differential equation if in 2010 ($t = 0$), the population of the more developed regions of the world was 1.2359×10^9.

(d) If the rate of growth continues to follow this model, what is the projected population of the more developed regions in 2020?

Source: *U.N. World Population Prospects*, 2010 update.

58. National Population Growth Barring disasters (human-made or natural), the population P of humans grows at a rate proportional to its current size. According to the U.N. World Population studies, from 2005 to 2010 the population of Ecuador grew at an annual rate of 1.490% per year. Assuming this growth rate continues:

(a) Write a differential equation that models the growth rate of the population.

(b) Find the general solution to the differential equation.

(c) Find the particular solution to the differential equation if in 2010 ($t = 0$), the population of Ecuador was 1.4465×10^7.

(d) If the rate of growth continues to follow this model, when will the projected population of Ecuador reach 20 million persons?

Source: *U.N. World Population Prospects*, 2010 update.

59. Oetzi the Iceman was found in 1991 by a German couple who were hiking in the Alps near the border of Austria and Italy. Carbon-14 testing determined that Oetzi died 5300 years ago. Assuming the half-life of carbon-14 is 5730 years, what percent of carbon-14 was left in his body? (An interesting note: In September 2010 the complete genome mapping of Oetzi was completed.)

60. Uninhibited Decay Radioactive beryllium is sometimes used to date fossils found in deep-sea sediments. (Carbon-14 dating cannot be used for fossils that lived underwater.) The decay of beryllium satisfies the equation $\dfrac{dA}{dt} = -\alpha A$, where $\alpha = 1.5 \times 10^{-7}$ and t is measured in years. What is the half-life of beryllium?

61. Decomposition of Sucrose Reacting with water in an acidic solution at 35 °C, sucrose ($C_{12}H_{22}O_{11}$) decomposes into glucose ($C_6H_{12}O_6$) and fructose ($C_6H_{12}O_6$) according to the law of uninhibited decay. An initial amount of 0.4 mol of sucrose decomposes to 0.36 mol in 30 min. How much sucrose will remain after 2 h? How long will it take until 0.10 mol of sucrose remains?

62. Chemical Dissociation Salt (NaCl) dissociates in water into sodium (Na^+) and chloride (Cl^-) ions at a rate proportional to its mass. The initial amount of salt is 25 kg, and after 10 h, 15 kg are left.

(a) How much salt will be left after 1 day?

(b) After how many hours will there be less than $\dfrac{1}{2}$ kg of salt left?

63. Voltage Drop The voltage of a certain condenser decreases at a rate proportional to the voltage. If the initial voltage is 20, and 2 s later it is 10, what is the voltage at time t? When will the voltage be 5?

64. Uninhibited Growth The rate of change in the number of bacteria in a culture is proportional to the number present. In a certain laboratory experiment, a culture has 10,000 bacteria initially, 20,000 bacteria at time t_1 minutes, and 100,000 bacteria at $(t_1 + 10)$ minutes.

(a) In terms of t only, find the number of bacteria in the culture at any time t minutes ($t \geq 0$).

(b) How many bacteria are there after 20 min?

(c) At what time are 20,000 bacteria observed? That is, find the value of t_1.

65. Verify that $\int x\sqrt{x}\,dx \neq \left(\int x\,dx\right)\left(\int \sqrt{x}\,dx\right)$.

66. Verify that $\int x(x^2+1)\,dx \neq x\int (x^2+1)\,dx$.

67. Verify that $\displaystyle\int \frac{x^2-1}{x-1}\,dx \neq \frac{\int (x^2-1)dx}{\int (x-1)\,dx}$.

68. Prove that $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

69. Derive the integration formula $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$, $a > 0$, $a \neq 1$. (*Hint:* Begin with the derivative of $y = a^x$.)

70. Use the formula from Problem 69 to find:

(a) $\int 2^x dx$ **(b)** $\int 3^x dx$

Challenge Problems

71. **(a)** Find y' if $y = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$.

(b) Use the result to show that

$$\int \sec x\, dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$

(c) Show that $\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| = \ln|\sec x + \tan x|$.

72. **(a)** Find y' if $y = x\sin^{-1}x + \sqrt{1-x^2}$.

(b) Use the result to show that

$$\int \sin^{-1} x\, dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

73. **(a)** Find y' if $y = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$.

(b) Use the result to show that

$$\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C$$

74. **(a)** Find y' if $y = \ln|\csc x - \cot x|$.

(b) Use the result to show that

$$\int \csc x\, dx = \ln|\csc x - \cot x| + C$$

75. Gudermannian Function

(a) Graph $y = \text{gd}(x) = \tan^{-1}(\sinh x)$. This function is called the **gudermannian of** x (named after Christoph Gudermann).

(b) If $y = \text{gd}(x)$, show that $\cos y = \text{sech}\, x$ and $\sin y = \tanh x$.

(c) Show that if $y = \text{gd}(x)$, then y satisfies the differential equation $y' = \cos y$.

(d) Use the differential equation of (c) to obtain the formula

$$\int \sec y\, dy = gd^{-1}(y) + C$$

Compare this to $\int \sec x\, dx = \ln|\sec x + \tan x| + C$.

76. The formula $\frac{d}{dx}\int f(x)\, dx = f(x)$ says that if a function is integrated and the result is differentiated, the original function is returned. What about the other way around? Is the formula $\int f'(x)\, dx = f(x)$ correct? Be sure to justify your answer.

5.6 Method of Substitution; Newton's Law of Cooling

OBJECTIVES *When you finish this section, you should be able to:*

1. **Find an indefinite integral using substitution (p. 45)**
2. **Find a definite integral using substitution (p. 49)**
3. **Integrate even and odd functions (p. 51)**
4. **Solve differential equations: Newton's Law of Cooling (p. 52)**

1 Find an Indefinite Integral Using Substitution

Indefinite integrals that cannot be found using the formulas in Table 1 on page 38 sometimes can be found using the *method of substitution.* In the method of substitution, we use a change of variables to transform the integrand so one of the formulas in the table applies.

NEED TO REVIEW? Differentials are discussed in Section 3.4, pp. xx–xx.

For example, to find $\int (x^2+5)^3 2x\, dx$, we use the substitution $u = x^2+5$. The differential of $u = x^2+5$ is $du = 2x\, dx$. Now we write $(x^2+5)^3 2x\, dx$ in terms of u and du, and integrate the simpler integral.

NOTE When using the method of substitution, once the substitution u is chosen, we write the original integral in terms of u and du.

$$\int \underbrace{(x^2+5)^3}_{u}\,\underbrace{2x\, dx}_{du} = \int u^3 du = \frac{u^4}{4} + C = \frac{(x^2+5)^4}{4} + C$$

$$\uparrow\ u = x^2+5$$

We can verify the answer by differentiating using the Power Rule for Functions.

$$\frac{d}{dx}\left[\frac{(x^2+5)^4}{4}+C\right]=\frac{1}{4}\left[4(x^2+5)^3(2x)\right]=(x^2+5)^3 2x$$

NEED TO REVIEW? The Chain Rule is discussed in Section 3.1, pp. xx–xx.

The method of substitution is based on the Chain Rule, which states that if f and g are differentiable functions, then for the composite function $f \circ g$,

$$\frac{d}{dx}(f\circ g)=\frac{d}{dx}f(g(x))=f'(g(x))\,g'(x)$$

The Chain Rule provides a template for finding integrals of the form

$$\int f'(g(x))g'(x)\,dx$$

If, in the integral, we let $u=g(x)$, then the differential $du=g'(x)\,dx$, and we have

$$\int f'(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du}=\int f'(u)\,du=f(u)+C=f(g(x))+C$$

Replacing $g(x)$ by u and $g'(x)dx$ by du is called **substitution**. Substitution is a strategy for finding antiderivatives when the integrand is a composite function.

EXAMPLE 1 Finding Indefinite Integrals Using Substitution

Find:

(a) $\displaystyle\int \sin(3x+2)\,dx$ **(b)** $\displaystyle\int x\sqrt{x^2+1}\,dx$ **(c)** $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

Solution **(a)** Since we know $\int \sin x\,dx$, we let $u=3x+2$. Then $du=3\,dx$ and $dx=\dfrac{du}{3}$.

$$\int \sin(\underbrace{3x+2}_{u})\,\underbrace{dx}_{\frac{du}{3}}=\int \sin u\,\frac{du}{3}=\frac{1}{3}\int \sin u\,du$$

$$=\frac{1}{3}(-\cos u)+C=-\frac{1}{3}\cos(3x+2)+C \qquad u=3x+2$$

(b) We let $u=x^2+1$. Then $du=2x\,dx$, so $x\,dx=\dfrac{du}{2}$.

$$\int x\sqrt{x^2+1}\,dx=\int \sqrt{\underbrace{x^2+1}_{u}}\,\underbrace{x\,dx}_{\frac{du}{2}}=\int \sqrt{u}\,\frac{du}{2}=\frac{1}{2}\int u^{1/2}du=\frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right)+C$$

$$=\frac{(x^2+1)^{3/2}}{3}+C$$

(c) We let $u=\sqrt{x}=x^{1/2}$. Then $du=\dfrac{1}{2}x^{-1/2}dx=\dfrac{dx}{2\sqrt{x}}$, so $\dfrac{dx}{\sqrt{x}}=2du$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx=\int e^{\sqrt{x}}\cdot\frac{dx}{\sqrt{x}}=\int e^u\cdot 2du=2e^u+C=2e^{\sqrt{x}}+C$$ ■

NOW WORK Problems 35 and 11.

When an integrand equals the product of an expression involving a function and its derivative (or a multiple of its derivative), then substitution is often a good strategy.

For example, for $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$, we used the substitution $u = \sqrt{x}$, since $\frac{du}{dx} = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ is a multiple of $\frac{1}{\sqrt{x}}$.

Similarly, in (b) the factor x in the integrand makes the substitution $u = x^2 + 1$ work. On the other hand, if we try to use this same substitution to integrate $\int \sqrt{x^2+1}\, dx$, then

$$\int \sqrt{x^2+1}\, dx = \int \sqrt{u}\, \frac{du}{2x} \underset{\uparrow}{=} \int \frac{\sqrt{u}}{2\sqrt{u-1}}\, du$$
$$x = \sqrt{u-1}$$

and the resulting integral is *more* complicated than the original integral.

The idea behind substitution is to obtain an integral $\int h(u)\, du$ that is simpler than the original integral $\int f(x)\, dx$. When a substitution does not simplify the integral, try other substitutions. If none of these work, other integration methods should be applied. Some of these methods are explored in Chapter 7.

EXAMPLE 2 Finding Indefinite Integrals Using Substitution

Find:

(a) $\int \frac{5x^2 dx}{4x^3 - 1}$ **(b)** $\int \frac{e^x}{e^x + 4} dx$

Solution **(a)** Notice that the numerator equals the derivative of the denominator, except for a constant factor. So, we try substitution. Let $u = 4x^3 - 1$. Then $du = 12x^2 dx$ and $5x^2 dx = \frac{5}{12}\, du$.

$$\int \frac{5x^2 dx}{4x^3 - 1} = \int \frac{\frac{5}{12}\, du}{u} = \frac{5}{12}\int \frac{du}{u} = \frac{5}{12}\ln|u| + C = \frac{5}{12}\ln\left|4x^3 - 1\right| + C$$

(b) Here, the numerator equals the derivative of the denominator. So, we use the substitution $u = e^x + 4$. Then $du = e^x dx$.

$$\int \frac{e^x}{e^x + 4} dx = \int \frac{1}{e^x + 4} \cdot e^x dx = \int \frac{1}{u}\, du = \ln|u| + C \underset{\uparrow}{=} \ln(e^x + 4) + C$$
$$u = e^x + 4 > 0$$

■

NOW WORK Problem 17.

EXAMPLE 3 Using Substitution To Establish an Integration Formula

Show that:

(a) $$\boxed{\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C}$$

(b) $$\boxed{\int \sec x\, dx = \ln|\sec x + \tan x| + C}$$

Solution **(a)** Since $\tan x = \frac{\sin x}{\cos x}$, we let $u = \cos x$. Then $du = -\sin x\, dx$ and

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

$$= \underset{\uparrow}{\ln|\cos x|^{-1}} + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$
$$r\ln x = \ln x^r$$

(b) To find $\int \sec x\, dx$, we multiply the integrand by $\dfrac{\sec x + \tan x}{\sec x + \tan x}$.

$$\int \sec x\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Now the numerator equals the derivative of the denominator. So if $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)dx$.

$$\int \sec x\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$ ■

NOW WORK **Problem 27.**

Examples 2 and 3 illustrate a basic integration formula:

$$\boxed{\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C} \tag{1}$$

IN WORDS If the numerator of the integrand equals the derivative of the denominator, then the integral equals a logarithmic function.

Notice that in formula (1) the integral equals the natural logarithm of the *absolute value of the function* g. The absolute value is necessary since the domain of the logarithm function is the set of positive real numbers. When g is known to be positive, as in Example 2(b), the absolute value is not required.

As we saw in Example 3(b), sometimes algebra is needed to transform an integral so that a basic integration formula can be used. Unlike differentiation, integration has no prescribed method; some ingenuity and a lot of practice are required. To illustrate, two different substitutions are used to solve Example 4.

EXAMPLE 4 Finding an Indefinite Integral Using Substitution

Find $\int x\sqrt{4+x}\, dx$.

Solution *Substitution I* Let $u = 4 + x$. Then $du = dx$. Since $u = 4 + x$, $x = u - 4$. Substituting gives

$$\int x\sqrt{4+x}\, dx \underset{u = x+4}{=} \int \underbrace{(u-4)}_{x}\ \underbrace{\sqrt{u}}_{4+x}\ \underbrace{du}_{dx} = \int (u^{3/2} - 4u^{1/2})\, du$$

$$= \frac{u^{5/2}}{\frac{5}{2}} - 4 \cdot \frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2(4+x)^{5/2}}{5} - \frac{8(4+x)^{3/2}}{3} + C$$

Substitution II Let $u = \sqrt{4+x}$, so $u^2 = 4 + x$ and $x = u^2 - 4$. Then $dx = 2u\, du$ and

$$\int x\sqrt{4+x}\, dx = \int (\underbrace{u^2-4}_{x})(u)(\underbrace{2u\, du}_{dx}) = 2\int (u^4 - 4u^2)\, du = 2\left[\frac{u^5}{5} - \frac{4u^3}{3}\right] + C$$

$$= \frac{2}{5}\left(\sqrt{4+x}\right)^5 - \frac{8}{3}\left(\sqrt{4+x}\right)^3 + C = \frac{2(4+x)^{5/2}}{5} - \frac{8(4+x)^{3/2}}{3} + C$$ ■

NOW WORK **Problem 35.**

EXAMPLE 5 **Finding Indefinite Integrals Using Substitution**

Find:

(a) $\int \frac{dx}{\sqrt{4-x^2}}$ **(b)** $\int \frac{dx}{9+4x^2}$

Solution **(a)** $\int \frac{dx}{\sqrt{4-x^2}}$ resembles $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$. We begin by rewriting the integrand as

$$\frac{1}{\sqrt{4-x^2}} = \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}} = \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

Now we let $u = \frac{x}{2}$. Then $du = \frac{dx}{2}$, so $dx = 2\,du$.

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} \underset{dx = 2\,du}{\overset{u = \frac{x}{2}}{=}} \int \frac{2\,du}{2\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C$$

(b) $\int \frac{dx}{9+4x^2}$ resembles $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$. We rewrite the integrand as

$$\frac{1}{9+4x^2} = \frac{1}{9\left(1+\frac{4x^2}{9}\right)} = \frac{1}{9\left[1+\left(\frac{2x}{3}\right)^2\right]}$$

Now let $u = \frac{2x}{3}$. Then $du = \frac{2}{3}dx$, so $dx = \frac{3}{2}\,du$.

$$\int \frac{dx}{9+4x^2} = \int \frac{dx}{9\left[1+\left(\frac{2x}{3}\right)^2\right]} = \int \frac{\frac{3}{2}du}{9(1+u^2)} = \frac{1}{6}\int \frac{du}{1+u^2}$$

$$= \frac{1}{6}\tan^{-1} u + C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C$$ ■

NOW WORK **Problem 39.**

2 Find a Definite Integral Using Substitution

Two approaches can be used to find a definite integral using substitution:

- Method 1: Find the related indefinite integral using substitution, and then apply the Fundamental Theorem of Calculus.
- Method 2: Find the definite integral directly by making a substitution in the integrand and using the substitution to *change the limits of integration.*

EXAMPLE 6 Finding a Definite Integral Using Substitution

Find $\int_0^2 x\sqrt{4-x^2}\,dx$.

Solution

Method 1: Use the related indefinite integral and then apply the Fundamental Theorem of Calculus. The related indefinite integral $\int x\sqrt{4-x^2}\,dx$ can be found using the substitution $u = 4 - x^2$. Then $du = -2x\,dx$ so $x\,dx = -\frac{du}{2}$.

$$\int x\sqrt{4-x^2}\,dx = \int \sqrt{u}\left(-\frac{du}{2}\right) = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\cdot\frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= -\frac{1}{3}(4-x^2)^{3/2} + C$$

Then by the Fundamental Theorem of Calculus,

$$\int_0^2 x\sqrt{4-x^2}\,dx = -\frac{1}{3}\left[(4-x^2)^{3/2}\right]_0^2 = -\frac{1}{3}\left[0 - 4^{3/2}\right] = \frac{8}{3}$$

Method 2: Find the definite integral directly by making a substitution in the integrand and changing the limits of integration. We let $u = 4 - x^2$; then $du = -2x\,dx$. Now use the function $u = 4 - x^2$ to change the limits of integration.

The lower limit of integration is $x = 0$ so, in terms of u, it becomes $u = 4 - 0^2 = 4$.

The upper limit of integration is $x = 2$ so the upper limit becomes $u = 4 - 2^2 = 0$.

Then

$$\int_0^2 x\sqrt{4-x^2}\,dx = \underset{\substack{\uparrow\\ u=4-x^2\\ xdx=-\frac{1}{2}du}}{\int_4^0} \sqrt{u}\left(-\frac{du}{2}\right) = -\frac{1}{2}\int_4^0 \sqrt{u}\,du = -\frac{1}{2}\cdot\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_4^0$$

$$= -\frac{1}{2}\left(0 - \frac{16}{3}\right) = \frac{8}{3}$$ ■

CAUTION When using substitution to find a definite integral directly, remember to change the limits of integration.

NOW WORK Problem 45.

EXAMPLE 7 Finding a Definite Integral Using Substitution

Find $\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\,d\theta$.

Solution We use properties of integrals to simplify before integrating.

$$\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}[1-\cos(2\theta)]\,d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2}\cos(2\theta)\,d\theta\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta - \frac{1}{2}\int_0^{\pi/2}\cos(2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\cos(2\theta)\,d\theta$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_0^{\pi/2}\cos(2\theta)\,d\theta$$

In the integral on the right, we use the substitution $u = 2\theta$. Then $du = 2\,d\theta$ and $d\theta = \dfrac{du}{2}$. Now we change the limits of integration:

$$\text{when } \theta = 0, \qquad \text{then } u = 2(0) = 0$$
$$\text{when } \theta = \frac{\pi}{2}, \qquad \text{then } u = 2\left(\frac{\pi}{2}\right) = \pi$$

Now

$$\int_0^{\pi/2} \cos(2\theta)\,d\theta = \int_0^{\pi} \cos u\,\frac{du}{2} = \frac{1}{2}\Big[\sin u\Big]_0^{\pi} = \frac{1}{2}(\sin\pi - \sin 0) = 0$$

Then,

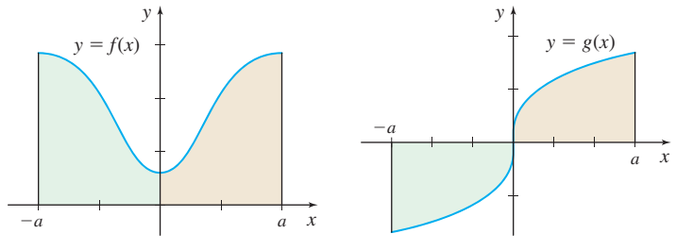
$$\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}d\theta = \frac{\pi}{4} - \frac{1}{2}\int_0^{\pi/2} \cos(2\theta)\,d\theta = \frac{\pi}{4}$$ ■

NOW WORK **Problem 53.**

3 Integrate Even and Odd Functions

NEED TO REVIEW? Even and odd functions are discussed in Section P.1, pp. xx–xx.

Integrals of even and odd functions can be simplified due to symmetry. Figure 32 illustrates the conclusions of the theorem that follows.



(a) f is even; $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ (b) g is odd; $\int_{-a}^{a} g(x)dx = 0$

Figure 32

THEOREM The Integrals of Even and Odd Functions

Let a function f be continuous on a closed interval $[-a, a]$, $a > 0$.

- If f is an even function, then

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

- If f is an odd function, then

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$

The property for even functions is proved here; the proof for odd functions is left as an exercise. See Problem 123.

Proof f is an even function: Since f is continuous on the closed interval $[-a, a]$, $a > 0$, and 0 is in the interval $[-a, a]$, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad (2)$$

In $-\int_0^{-a} f(x)\,dx$, we use the substitution $u = -x$. Then $du = -dx$. Also, if $x = 0$, then $u = 0$, and if $x = -a$, then $u = a$. Therefore,

$$-\int_{0}^{-a} f(x)\,dx = \int_{0}^{a} f(-u)\,du = \int_{0}^{a} f(u)du = \int_{0}^{a} f(x)\,dx \qquad (3)$$

(f is even: $f(-u) = f(u)$)

Combining (2) and (3), we obtain

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$ ■

To use the theorem involving even and odd functions, three conditions must be met:

- The function f must be even or odd.
- The function f must be continuous on the closed interval $[-a,\ a]$, $a > 0$.
- The limits of integration must be $-a$ and a, $a > 0$.

EXAMPLE 8 Integrating an Even or Odd Function

Find:

(a) $\int_{-3}^{3} (x^7 - 4x^3 + x)\,dx$ **(b)** $\int_{-2}^{2} (x^4 - x^2 + 3)\,dx$

Solution **(a)** If $f(x) = x^7 - 4x^3 + x$, then $f(-x) = (-x)^7 - 4(-x)^3 + (-x) = -(x^7 - 4x^3 + x) = -f(x)$. Since f is an odd function,

$$\int_{-3}^{3} (x^7 - 4x^3 + x)\,dx = 0$$

(b) If $g(x) = x^4 - x^2 + 3$, then $g(-x) = (-x)^4 - (-x)^2 + 3 = x^4 - x^2 + 3 = g(x)$. Since g is an even function,

$$\int_{-2}^{2} (x^4 - x^2 + 3)dx = 2\int_{0}^{2} (x^4 - x^2 + 3)\,dx = 2\left[\frac{x^5}{5} - \frac{x^3}{3} + 3x\right]_0^2$$

$$= 2\left[\frac{32}{5} - \frac{8}{3} + 6\right] = \frac{292}{15}$$ ■

NOW WORK Problems 63 and 67.

EXAMPLE 9 Using Properties of Integrals

If f is an even function and $\int_0^2 f(x)\,dx = -6$ and $\int_{-5}^{0} f(x)\,dx = 8$, find $\int_2^5 f(x)\,dx$.

Solution $\int_2^5 f(x)\,dx = \int_2^0 f(x)\,dx + \int_0^5 f(x)\,dx$

Now $\int_2^0 f(x)\,dx = -\int_0^2 f(x)\,dx = 6$.

Since f is even, $\int_0^5 f(x)\,dx = \int_{-5}^{0} f(x)\,dx = 8$. Then

$$\int_2^5 f(x)\,dx = \int_2^0 f(x)\,dx + \int_0^5 f(x)\,dx = 6 + 8 = 14$$ ■

4 Solve Differential Equations: Newton's Law of Cooling

Suppose an object is heated to a temperature u_0. Then at time $t = 0$, the object is put into a medium with a constant lower temperature causing the object to cool. Newton's Law of Cooling states that the rate of change of the temperature of the object with respect to time is continuous and proportional to the difference between the temperature of the object and the ambient temperature (the temperature of the surrounding medium). That is, if $u = u(t)$ is the temperature of the object at time t and if T is the (constant) ambient temperature, then Newton's Law of Cooling is modeled by the differential equation

$$\boxed{\frac{du}{dt} = k[u(t) - T]} \tag{4}$$

where k is a constant that depends on the object. Since the ambient temperature T is lower than $u(0) = u_0$, the object cools and its temperature decreases so that $\frac{du}{dt} < 0$. Then, since $u(t) > T$, k is a negative constant.

We find u as a function of t by solving the differential equation $\frac{du}{dt} = k\,(u - T)$. We rewrite the differential equation as $\frac{du}{u - T} = k\,dt$ and integrate both sides.

$$\int \frac{du}{u - T} = \int k\,dt$$

$$\ln|u - T| = k\,t + C$$

To find C, we use the boundary condition that at time $t = 0$, the initial temperature of the object is $u(0) = u_0$, Then

$$\ln|u_0 - T| = k \cdot 0 + C$$

$$C = \ln|u_0 - T|$$

Using this expression for C, we obtain

$$\ln|u - T| = k\,t + \ln|u_0 - T|$$

$$\ln|u - T| - \ln|u_0 - T| = k\,t$$

$$\ln\left|\frac{u - T}{u_0 - T}\right| = k\,t$$

$$\frac{u - T}{u_0 - T} = e^{kt}$$

$$u - T = (u_0 - T)e^{kt}$$

$$\boxed{u = (u_0 - T)\,e^{kt} + T} \tag{5}$$

EXAMPLE 10 Using Newton's Law of Cooling

An object is heated to 90 °C and allowed to cool in a room with a constant ambient temperature of 20 °C. If after 10 min the temperature of the object is 60 °C, what will its temperature be after 20 min?

Solution When $t = 0$, $u(0) = u_0 = 90$ °C, and when $t = 10$ min, $u\,(10) = 60$ °C. Given that the ambient temperature T is 20 °C, we substitute these values into equation (5).

$$u(t) = (u_0 - T)e^{kt} + T$$

$$60 = (90 - 20)e^{10k} + 20 \qquad u = 60 \text{ when } t = 10;\ T = 20;\ u_0 = 90$$

$$\frac{40}{70} = e^{10k}$$

$$k = \frac{1}{10}\ln\frac{4}{7} = 0.10\ln\frac{4}{7}$$

The temperature u is

$$u(t) = 70e^{0.1\ln(4/7)} + 20$$

Then when $t = 20$, the temperature u of the object is

$$u(20) = 70e0.1\ln(4/7) + 20 = 70e^{2\ln 4/7} + 20 \approx 42.86\,°\text{C}$$ ■

NOW WORK Problem 109.

5.6 Assess Your Understanding

Concepts and Vocabulary

1. If the substitution $u = 2x + 3$ is used with $\int \sin(2x + 3)\,dx$, the result is $\int$ _____ du.
2. *True or False* If the substitution $u = x^2 + 3$ is used with $\int_0^1 x(x^2+3)^3 dx$, the result is $\int_0^1 x(x^2+3)^3 dx = \frac{1}{2}\int_0^1 u^3 du$.
3. *Multiple Choice* $\int_{-4}^{4} x^3 dx =$ [(**a**) 128 (**b**) 4 (**c**) 0 (**d**) 64].
4. *True or False* $\int_0^5 x^2 dx = \frac{1}{2}\int_{-5}^{5} x^2 dx$.

Skill Building

In Problems 5–10, find each indefinite integral using the given substitution.

5. $\int e^{3x+1} dx$; let $u = 3x + 1$.
6. $\int \frac{dx}{x \ln x}$; let $u = \ln x$.
7. $\int (1 - t^2)^6 t\,dt$; let $u = 1 - t^2$.
8. $\int \sin^5 x \cos x\,dx$; let $u = \sin x$.
9. $\int \frac{x^2\,dx}{\sqrt{1 - x^6}}$; let $u = x^3$.
10. $\int \frac{e^{-x}}{6 + e^{-x}} dx$; let $u = 6 + e^{-x}$.

In Problems 11–44, find each indefinite integral.

11. $\int \sin(3x)\,dx$
12. $\int x \sin x^2\,dx$
13. $\int \sin x \cos^2 x\,dx$
14. $\int \tan^2 x \sec^2 x\,dx$
15. $\int \frac{e^{1/x}}{x^2} dx$
16. $\int \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$
17. $\int \frac{x\,dx}{x^2 - 1}$
18. $\int \frac{5x\,dx}{1 - x^2}$
19. $\int \frac{e^x}{\sqrt{1 + e^x}} dx$
20. $\int \frac{dx}{x(\ln x)^7}$
21. $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^4} dx$
22. $\int \frac{dx}{\sqrt{x}(1 + \sqrt{x})}$
23. $\int \frac{3e^x}{\sqrt[4]{e^x - 1}} dx$
24. $\int \frac{[\ln(5x)]^3}{x} dx$
25. $\int \frac{\cos x\,dx}{2\sin x - 1}$
26. $\int \frac{\cos(2x)\,dx}{\sin(2x)}$
27. $\int \sec(5x)\,dx$
28. $\int \tan(2x)\,dx$
29. $\int \sqrt{\tan x} \sec^2 x\,dx$
30. $\int (2 + 3\cot x)^{3/2} \csc^2 x\,dx$
31. $\int \frac{\sin x}{\cos^2 x} dx$
32. $\int \frac{\cos x}{\sin^2 x} dx$
33. $\int \sin x \cdot e^{\cos x}\,dx$
34. $\int \sec^2 x \cdot e^{\tan x}\,dx$
35. $\int x\sqrt{x + 3}\,dx$
36. $\int x\sqrt{4 - x}\,dx$
37. $\int [\sin x + \cos(3x)]dx$
38. $\int \left[x^2 + \sqrt{3x + 2}\right] dx$
39. $\int \frac{dx}{x^2 + 25}$
40. $\int \frac{\cos x}{1 + \sin^2 x} dx$
41. $\int \frac{dx}{\sqrt{9 - x^2}}$
42. $\int \frac{dx}{\sqrt{16 - 9x^2}}$
43. $\int \sinh x \cosh x\,dx$
44. $\int \operatorname{sech}^2 x \tanh x\,dx$

In Problems 45–52, find each definite integral two ways:

(a) *By finding the related indefinite integral and then using the Fundamental Theorem of Calculus.*

(b) *By making a substitution in the integrand and using the substitution to change the limits of integration.*

(c) *Which method did you prefer? Why?*

45. $\int_{-2}^{0} \frac{x}{(x^2 + 3)^2} dx$
46. $\int_{-1}^{1} (s^2 - 1)^5 s\,ds$
47. $\int_0^1 x^2 e^{x^3+1} dx$
48. $\int_0^1 x e^{x^2-2} dx$
49. $\int_1^6 x\sqrt{x + 3}\,dx$
50. $\int_2^6 x^2\sqrt{x - 2}\,dx$
51. $\int_0^2 x \cdot 3^{2x^2} dx$
52. $\int_0^1 x \cdot 10^{-x^2} dx$

In Problems 53–62, find each definite integral.

53. $\int_1^3 \frac{1}{x^2}\sqrt{1 - \frac{1}{x}}\,dx$
54. $\int_0^{\pi/4} \frac{\sin(2x)}{\sqrt{5 - 2\cos(2x)}} dx$
55. $\int_0^2 \frac{e^{2x}}{e^{2x} + 1} dx$
56. $\int_1^3 \frac{e^{3x}}{e^{3x} - 1} dx$
57. $\int_2^3 \frac{dx}{x \ln x}$
58. $\int_2^3 \frac{dx}{x(\ln x)^2}$
59. $\int_0^{\pi} e^x \cos(e^x)\,dx$
60. $\int_0^{\pi} e^{-x} \cos(e^{-x})\,dx$
61. $\int_0^1 \frac{x\,dx}{1 + x^4}$
62. $\int_0^1 \frac{e^x}{1 + e^{2x}} dx$

In Problems 63–70, use properties of integrals to find each integral.

63. $\int_{-2}^{2} (x^2 - 4)\,dx$
64. $\int_{-1}^{1} (x^3 - 2x)\,dx$

65. $\int_{-\pi/2}^{\pi/2} \frac{1}{3}\sin\theta d\theta$

66. $\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

67. $\int_{-\pi/4}^{\pi/4} \frac{3}{1+x^2}\, dx$

68. $\int_{-\pi/4}^{\pi/4} \sin x\, dx$

69. $\int_{-5}^{5} |2x|\, dx$

70. $\int_{-1}^{1} [|x| - 3]\, dx$

Applications and Extensions

In Problems 71–84, find each integral.

71. $\int \frac{x+1}{x^2+1} dx$

72. $\int \frac{2x-3}{1+x^2} dx$

73. $\int \left(2\sqrt{x^2+3} - \frac{4}{x} + 9\right)^6 \left(\frac{x}{\sqrt{x^2+3}} + \frac{2}{x^2}\right) dx$

74. $\int \left[\sqrt{(z^2+1)^4 - 3}\right]\left[z(z^2+1)^3\right] dz$

75. $\int \frac{x+4x^3}{\sqrt{x}}\, dx$

76. $\int \frac{z\, dz}{z+\sqrt{z^2+4}}$

77. $\int \sqrt{t}\sqrt{4+t\sqrt{t}}\, dt$

78. $\int_0^1 \frac{x+1}{x^2+3} dx$

79. $\int 3^{2x+1} dx$

80. $\int 2^{3x+5} dx$

81. $\int \frac{\sin x}{\sqrt{4-\cos^2 x}} dx$

82. $\int \frac{\sec^2 x\, dx}{\sqrt{1-\tan^2 x}}$

83. $\int_0^1 \frac{(z^2+5)(z^3+15z-3)}{196-(z^3+15z-3)^2} dz$

84. $\int_2^{17} \frac{dx}{\sqrt{\sqrt{x-1}+(x-1)^{5/4}}}$

In Problems 85–90, find each integral. (Hint: Begin by using a Change of Base formula.)

85. $\int \frac{dx}{x\log_{10} x}$

86. $\int \frac{dx}{x\log_3 \sqrt[5]{x}}$

87. $\int_{10}^{100} \frac{dx}{x\log x}$

88. $\int_{10}^{100} \frac{dx}{x\ln x}$

89. $\int_3^9 \frac{dx}{x\log_3 x}$

90. $\int_{10}^{100} \frac{dx}{x\log_5 x}$

91. If $\int_1^b t^2(5t^3-1)^{1/2}\, dt = \frac{38}{45}$, find b.

92. If $\int_a^3 t\sqrt{9-t^2}\, dt = 6$, find a.

In Problems 93 and 94, find each indefinite integral by:
(a) *First using substitution.*
(b) *First expanding the integrand.*
(c) *Compare the two results.*

93. $\int (x+1)^2\, dx$

94. $\int (x^2+1)^2 x\, dx$

In Problems 95 and 96, find each integral three ways:
(a) *By using substitution.*
(b) *By using properties of the definite integral.*
(c) *By using trigonometry to simplify the integrand before integrating.*
(d) *Compare the results.*

95. $\int_{-\pi/2}^{\pi/2} \cos(2x+\pi)\, dx$

96. $\int_{-\pi/4}^{\pi/4} \sin(7\theta-\pi)\, d\theta$

97. Area Find the area under the graph of $f(x) = \frac{x^2}{\sqrt{2x+1}}$ from 0 to 4.

98. Area Find the area under the graph of $f(x) = \frac{x}{(x^2+1)^2}$ from 0 to 2.

99. Area Find the area under the graph of $y = \frac{1}{3x^2+1}$ from $x = 0$ to $x = 1$.

100. Area Find the area under the graph of $y = \frac{1}{x\sqrt{x^2-4}}$ from $x = 3$ to $x = 4$.

101. Area Find the area under the graph of the catenary,

$$y = a\cosh\frac{x}{a} + b - a,$$

from $x = 0$ to $x = a$.

102. Area Find b so that the area under the graph of

$$y = (x+1)\sqrt{x^2+2x+4}$$

is $\frac{56}{3}$ for $0 \le x \le b$.

103. Average Value Find the average value of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

104. Average Value Find the average value of $y = \sec x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

105. If $\int_0^2 f(x-3)\, dx = 8$, find $\int_{-3}^{-1} f(x)\, dx$.

106. If $\int_{-2}^{1} f(x+1)\, dx = \frac{5}{2}$, find $\int_{-1}^{2} f(x)\, dx$.

107. If $\int_0^4 f\left(\frac{x}{2}\right) dx = 8$, find $\int_0^2 f(x)\, dx$.

108. If $\int_0^1 g(3x)\, dx = 6$, find $\int_0^3 g(x)\, dx$.

109. Newton's Law of Cooling Newton's Law of Cooling states that the rate of change of temperature with respect to time is proportional to the difference between the temperature of the object and the ambient temperature. A thermometer that reads 4 °C is brought into a room that is 30 °C.

(a) Write the differential equation that models the temperature $u = u(t)$ of the thermometer at time t in minutes (min).

(b) Find the general solution of the differential equation.

(c) If the thermometer reads 10 °C after 2 min, determine the temperature reading 5 min after the thermometer is first brought into the room.

110. Newton's Law of Cooling A thermometer reading 70 °F is taken outside where the ambient temperature is 22 °F. Four minutes later the reading is 32 °F.

(a) Write the differential equation that models the temperature $u = u(t)$ of the thermometer at time t.

(b) Find the general solution of the differential equation.

(c) Find the particular solution to the differential equation, using the initial condition that when $t = 0$ min, then $u = 70$ °F.

(d) Find the thermometer reading 7 min after the thermometer was brought outside.

(e) Find the time it takes for the reading to change from 70 °F to within $\frac{1}{2}$ °F of the air temperature.

111. Forensic Science At 4 p.m., a body was found floating in water whose temperature is 12 °C. When the woman was alive, her body temperature was 37 °C and now it is 20 °C. Suppose the rate of change of the temperature $u = u(t)$ of the body with respect to the time t in hours (h) is proportional to $u(t) - T$, where T is the water temperature and the constant of proportionality is -0.159.

(a) Write a differential equation that models the temperature $u = u(t)$ of the body at time t.

(b) Find the general solution of the differential equation.

(c) Find the particular solution to the differential equation, using the initial condition that at the time of death, when $t = 0$ h, her body temperature was $u = 37$ °C.

(d) At what time did the woman drown?

(e) Will the woman's body ever cool to 12°C? Explain.

112. Newton's Law of Cooling A pie is removed from a 350 °F oven to cool in a room whose temperature is 72 °F.

(a) Write the differential equation that models the temperature $u = u(t)$ of the pie at time t.

(b) Find the general solution of the differential equation.

(c) Find the particular solution to the differential equation, using the initial condition that when $t = 0$ min, then $u = 350$ °F.

(d) If $u(5) = 200$ °F, what is the temperature of the pie after 15 min?

(e) How long will it take for the pie to be 100 °F and ready to eat?

113. Electric Potential The electric field strength a distance z from the axis of a ring of radius R carrying a charge Q is given by the formula

$$E(z) = \frac{Qz}{(R^2 + z^2)^{3/2}}$$

If the electric potential V is related to E by $E = -\dfrac{dV}{dz}$, what is $V(z)$?

114. Impulse During a Rocket Launch The impulse J due to a force F is the product of the force times the amount of time t for which the force acts. When the force varies over time, $J = \int_{t_1}^{t_2} F(t)\,dt$. We can model the force acting on a rocket during launch by an exponential function $F(t) = Ae^{bt}$, where A and b are constants that depend on the characteristics of the engine. At the instant lift-off occurs ($t = 0$), the force must equal the weight of the rocket.

(a) Suppose the rocket weighs 25,000 N (a mass of about 2500 kg or a weight of 5500 lb), and 30 seconds after lift-off the force acting on the rocket equals twice the weight of the rocket. Find A and b.

(b) Find the impulse delivered to the rocket during the first 30 seconds after the launch.

115. Air Resistance on a Falling Object If an object of mass m is dropped, the air resistance on it when it has speed v can be modeled as $F_{\text{air}} = -kv$, where the constant k depends on the shape of the object and the condition of the air. The minus sign is necessary because the direction of the force is opposite to the velocity. Using Newton's Second Law of Motion, this force leads to a downward acceleration $a(t) = ge^{-kt/m}$. See Problem 137. Using the equation for $a(t)$, find:

(a) $v(t)$, if the object starts from rest $v_0 = v(0) = 0$, with the positive direction downward.

(b) $s(t)$, if the object starts from the position $s_0 = s(0) = 0$, with the positive direction downward.

(c) What limits do $a(t)$, $v(t)$, and $s(t)$ approach if the object falls for a very long time ($t \to \infty$)? Interpret each result and explain if it is physically reasonable.

(d) Graph $a = a(t)$, $v = v(t)$, $s = s(t)$. Do the graphs support the conclusions obtained in part (c)?

116. Area Let $f(x) = k\sin(kx)$, where k is a positive constant.

(a) Find the area of the region under one arch of the graph of f.

(b) Find the area of the triangle formed by the x-axis and the tangent lines to one arch of f at the points where the graph of f crosses the x-axis.

117. Use an appropriate substitution to show that

$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx,$$

where m, n are positive integers.

118. Properties of Integrals Find $\int_{-1}^{1} f(x)\,dx$ for the function given below:

$$f(x) = \begin{cases} x + 1 & \text{if} \quad x < 0 \\ \cos(\pi x) & \text{if} \quad x \geq 0 \end{cases}$$

119. If f is continuous on $[a, b]$, show that

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

120. If $\int_0^1 f(x)\,dx = 2$, find:

(a) $\int_0^{0.5} f(2x)\,dx$ **(b)** $\int_0^3 f\left(\frac{1}{3}x\right)dx$

(c) $\int_0^{1/5} f(5x)\,dx$

(d) Find the upper and lower limits of integration so that

$$\int_a^b f\left(\frac{x}{4}\right)dx = 8.$$

(e) Generalize (d) so that $\int_a^b f(kx)dx = \frac{1}{k}\cdot 2$ for $k > 0$.

121. If $\int_0^2 f(s)\,ds = 5$, find:

(a) $\int_{-1}^1 f(s+1)\,ds$ **(b)** $\int_{-3}^{-1} f(s+3)\,ds$

(c) $\int_4^6 f(s-4)\,ds$

(d) Find the upper and lower limits of integration so that

$$\int_a^b f(s-2)ds = 5.$$

(e) Generalize (d) so that $\int_a^b f(s-k)\,ds = 5$ for $k > 0$.

122. Find $\int_0^b |2x|\,dx$ for any real number b.

123. If f is an odd function, show that $\int_{-a}^a f(x)\,dx = 0$.

124. Find the constant k, where $0 \le k \le 3$, for which

$$\int_0^3 \frac{x}{\sqrt{x^2+16}}dx = \frac{3k}{\sqrt{k^2+16}}$$

125. If n is a nonnegative integer, for what n does

$$\int_0^1 x^n\,dx = \int_0^1 (1-x)^n\,dx?$$

126. If f is a continuous function defined on the interval $[0, 1]$, show that

$$\int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

127. Prove that $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.
[*Hint:* Multiply and divide the integrand by $(\csc x - \cot x)$.]

128. Describe a method for finding $\int_a^b |f(x)|\,dx$ in terms of $F(x) = \int f(x)\,dx$ when $f(x)$ has finitely many zeros.

129. Find $\int \sqrt[n]{a+bx}\,dx$, where a and b are real numbers and $b \ne 0$.

130. If f'' is continuous on (a, b), show that

$$\int_a^b xf''(x)\,dx = bf'(b) - af'(a) - f(b) + f(a)$$

[*Hint*: Look at the derivative of $F(x) = xf'(x) - f(x)$.]

131. If f is continuous for all x, which of the following integrals have the same value?

(a) $\int_a^b f(x)\,dx$ **(b)** $\int_0^{b-a} f(x+a)\,dx$

(c) $\int_{a+c}^{b+c} f(x+c)\,dx$

Challenge Problems

132. Find $\int \frac{x^6+3x^4+3x^2+x+1}{(x^2+1)^2}\,dx$.

133. Find $\int \frac{\sqrt[4]{x}}{\sqrt{x}+\sqrt[3]{x}}\,dx$.

134. Find $\int \frac{3x+2}{x\sqrt{x+1}}\,dx$

135. Find $\int \frac{dx}{(x\ln x)[\ln(\ln x)]}$

136. Suppose that the graph of $y = f(x)$ contains the points $(0, 1)$ and $(2, 5)$. Find $\int_0^2 f'(x)\,dx$. (Assume that f' is continuous.)

137. Air Resistance on a Falling Object (Refer to Problem 115.) If an object of mass m is dropped, the air resistance on it when it has speed v can be modeled as $F_{\text{air}} = -kv$, where the constant k depends on the shape of the object and the condition of the air. The minus sign is necessary because the direction of the force is opposite to the velocity. Using Newton's Second Law of Motion, show that the downward acceleration of the object is $a(t) = ge^{-kt/m}$, where g is the acceleration due to gravity. (*Hint:* The velocity of the object obeys the differential equation

$$m\frac{dv}{dt} = mg - kv$$

Solve the differential equation for v and use the fact that $ma = mg - kv$.)

138. A **separable differential equation** can be written in the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$, where f and g are continuous. Then

$$\int g(y)\,dy = \int f(x)\,dx$$

and integrating (if possible) will give a solution to the differential equation. Use this technique to solve parts (a)–(c) below. (You may need to leave your answer in implicit form.)

(a) $\frac{y^2}{x}\frac{dy}{dx} = 1 + x^2$ **(b)** $\frac{dy}{dx} = y\frac{x^2-2x+1}{y+3}$

(c) $y\frac{dy}{dx} = \frac{x^2}{y+4}$; if $y = 2$ when $x = 8$

Chapter Review

THINGS TO KNOW

5.1 Area

Definitions:

- Partition of an interval $[a, b]$ (p. 3)
- Area A under the graph of a function f from a to b (p. 6)

5.2 The Definite Integral

Definitions:

- Riemann sums (p. 12)
- The definite integral (p. 13)
- $\displaystyle\int_a^a f(x)\,dx = 0$ (p. 14)
- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (p. 14)

Theorem: If a function f is continuous on a closed interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ exists. (p. 14)

- $\displaystyle\int_a^b h\,dx = h(b-a)$, h a constant (p. 15)

5.3 The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus: Let f be a function that is continuous on a closed interval $[a, b]$.

- Part 1: The function I defined by $I(x) = \int_a^x f(t)\,dt$ has the properties that it is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $I'(x) = \dfrac{d}{dx}\left[\displaystyle\int_a^x f(t)\,dt\right] = f(x)$, for all x in (a, b). (p. 20)
- Part 2: If F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$. (p. 22)

5.4 Properties of the Definite Integral

Properties of definite integrals:

If two functions f and g are continuous on the closed interval $[a, b]$ and k is a constant, then

- Integral of a sum:
$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \quad \text{(p. 27)}$$
- Integral of a constant times a function:
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \quad \text{(p. 27)}$$
- $\displaystyle\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\,dx =$
$k_1\displaystyle\int_a^b f_1(x)\,dx + k_2\int_a^b f_2(x)\,dx + \cdots + k_n\int_a^b f_n(x)\,dx.$ (p. 28)
- If f is continuous on an interval containing the numbers a, b, and c, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \text{(p. 28)}$$
- *Bounds on an Integral:* If a function f is continuous on a closed interval $[a, b]$ and if m and M denote the absolute minimum and absolute maximum values, respectively, of f on $[a, b]$, then
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \quad \text{(p. 29)}$$
- *Mean Value Theorem for Integrals:* If a function f is continuous on a closed interval $[a, b]$, then there is a real number u, where $a \le u \le b$, for which
$$\int_a^b f(x)\,dx = f(u)(b-a). \quad \text{(p. 30)}$$

Definition: The average value of a function over an interval $[a, b]$ is
$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx. \text{ (p. 31)}$$

5.5 The Indefinite Integral; Growth and Decay Models

The indefinite integral of f: $\int f(x)\,dx = F(x) + C$ if and only if $\dfrac{d}{dx}[F(x) + C] = f(x)$, where C is the constant of integration. (p. 37)

Basic integration formulas: See Table 1. (p. 38)

Properties of indefinite integrals:

- Derivative of an integral:
$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x) \text{ (p. 38)}$$
- Integral of a sum:
$$\int [f(x) + g(x)]dx = \int f(x)\,dx + \int g(x)dx \quad \text{(p. 39)}$$
- Integral of a constant times a function:
$$\int k\,f(x)\,dx = k\int f(x)\,dx \quad \text{(p. 39)}$$

5.6 Method of Substitution; Newton's Law of Cooling

Method of substitution: (p. 46)

Method of substitution (definite integrals):

- Find the related indefinite integral using substitution. Then use the Fundamental Theorem of Calculus. (p. 49)
- Find the definite integral directly by making a substitution in the integrand and using the substitution to change the limits of integration. (p. 49)

Basic integration formulas:

- $\displaystyle\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C \quad \text{(p. 48)}$
- If f is an even function, then
$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx. \quad \text{(p. 51)}$$
- If f is an odd function, then $\displaystyle\int_{-a}^a f(x)\,dx = 0.$ (p. 51)

OBJECTIVES

Section	You should be able to …	Examples	Review Exercises
5.1	1 Approximate the area under the graph of a function (p. 2)	1, 2	1, 2
	2 Find the area under the graph of a function (p. 6)	3, 4	3, 4
5.2	1 Define a definite integral as the limit of Riemann sums (p. 11)	1, 2	5(a), (b)
	2 Find a definite integral using the limit of Riemann sums (p. 14)	3–5	5(c), 6
5.3	1 Use Part 1 of the Fundamental Theorem of Calculus (p. 21)	1–3	7-10, 52, 53
	2 Use Part 2 of the Fundamental Theorem of Calculus (p. 23)	4, 5	5(d), 11–13, 15–18, 47, 56
	3 Interpret an integral using Part 2 of the Fundamental Theorem of Calculus (p. 23)	6	21, 22, 57
5.4	1 Use properties of the definite integral (p. 27)	1–6	23, 24, 27, 28, 53
	2 Work with the Mean Value Theorem for Integrals (p. 30)	7	29, 30
	3 Find the average value of a function (p. 31)	8	31–34
5.5	1 Find indefinite integrals (p. 37)	1	14
	2 Use properties of indefinite integrals (p. 38)	2, 3	19, 20, 35, 36
	3 Solve differential equations involving growth and decay (p. 40)	4, 5	37, 38, 59, 60
5.6	1 Find an indefinite integral using substitution (p. 45)	1–5	39–41, 44, 45, 48, 51
	2 Find a definite integral using substitution (p. 49)	6, 7	42, 43, 46, 47, 50, 46–48, 51, 52, 55, 56, 61, 62
	3 Integrate even and odd functions (p. 51)	8, 9	25–28
	4 Solve differential equations: Newton's Law of Cooling (p. 52)	10	65

REVIEW EXERCISES

1. Area Approximate the area under the graph of $f(x) = 2x + 1$ from 0 to 4 by finding s_n and S_n for $n = 4$ and $n = 8$.

2. Area Approximate the area under the graph of $f(x) = x^2$ from 0 to 8 by finding s_n and S_n for $n = 4$ and $n = 8$ subintervals.

3. Area Find the area A under the graph of $y = f(x) = 9 - x^2$ from 0 to 3 by using lower sums s_n (rectangles that lie below the graph of f).

4. Area Find the area A under the graph of $y = f(x) = 8 - 2x$ from 0 to 4 using upper sums S_n (rectangles that lie above the graph of f).

5. Riemann Sums

(a) Find the Riemann sum of $f(x) = x^2 - 3x + 3$ on the closed interval $[-1, 3]$ using a regular partition with four subintervals and the numbers $u_1 = -1$, $u_2 = 0$, $u_3 = 2$, and $u_4 = 3$.

(b) Find the Riemann sums of f by partitioning $[-1, 3]$ into n subintervals of equal length and choosing u_i as the right endpoint of the ith subinterval $[x_{i-1}, x_i]$. Write the limit of the Riemann sums as a definite integral. Do not evaluate.

(c) Find the limit as n approaches ∞ of the Riemann sums found in (b).

(d) Find the definite integral from (b) using the Fundamental Theorem of Calculus. Compare the answer to the limit found in (c).

6. Units of an Integral In the definite integral $\int_a^b a(t)\,dt$, where a represents acceleration measured in meters per second squared and t is measured in seconds, what are the units of $\int_a^b a(t)\,dt$?

In Problems 7–10, find each derivative using the Fundamental Theorem of Calculus.

7. $\dfrac{d}{dx}\displaystyle\int_0^x t^{2/3}\sin t\,dt$

8. $\dfrac{d}{dx}\displaystyle\int_e^x \ln t\,dt$

9. $\dfrac{d}{dx}\displaystyle\int_{x^2}^1 \tan t\,dt$

10. $\dfrac{d}{dx}\displaystyle\int_a^{2\sqrt{x}} \dfrac{t}{t^2+1}\,dt$

In Problems 11–20, find each integral.

11. $\displaystyle\int_1^{\sqrt{2}} x^{-2}\,dx$

12. $\displaystyle\int_1^{e^2} \frac{1}{x}\,dx$

13. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

14. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx$

15. $\displaystyle\int_0^{\ln 2} 4e^x\,dx$

16. $\displaystyle\int_0^2 (x^2 - 3x + 2)\,dx$

17. $\displaystyle\int_1^4 2^x dx$

18. $\displaystyle\int_0^{\pi/4} \sec x\,\tan x\,dx$

19. $\displaystyle\int \left(\frac{1+2xe^x}{x}\right)dx$

20. $\displaystyle\int \frac{1}{2}\sin x\,dx$

21. Interpreting an Integral The function $v = v(t)$ is the speed v, in kilometers per hour, of a train at a time t, in hours. Interpret the integral $\int_0^{16} v(t)\,dt = 460$.

22. Interpreting an Integral The function $V = f(t)$ is the volume V of oil, in liters per hour, draining from a storage tank at time t (in hours). Interpret the integral $\int_0^2 f(t)dt = 100$.

In Problems 23–26, find each integral.

23. $\displaystyle\int_{-2}^{2} f(x)\,dx$, where $f(x) = \begin{cases} 3x+2 & \text{if } -2 \le x < 0 \\ 2x^2+2 & \text{if } 0 \le x \le 2 \end{cases}$

24. $\displaystyle\int_{-1}^{4} |x|\,dx$

25. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,dx$

26. $\displaystyle\int_{-3}^{3} \frac{x^2}{x^2+9}\,dx$

Bounds on an Integral *In Problems 27 and 28, find lower and upper bounds for each integral.*

27. $\displaystyle\int_{-4}^{-1} (2x^3+9x^2+12x+32)\,dx$

28. $\displaystyle\int_{0}^{3} x^2(x^2-1)^{1/3}\,dx$

In Problems 29 and 30, for each integral find the number(s) u guaranteed by the Mean Value Theorem for Integrals.

29. $\displaystyle\int_{0}^{\pi} \sin x\,dx$

30. $\displaystyle\int_{-3}^{3} (x^3+2x)dx$

In Problems 31–34, find the average value of each function over the given interval.

31. $f(x) = \sin x$ over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

32. $f(x) = x^3$ over $[1, 4]$

33. $f(x) = e^x$ over $[-1, 1]$

34. $f(x) = 6x^{2/3}$ over $[0, 8]$

35. Find $\dfrac{d}{dx}\displaystyle\int \sqrt{\frac{1}{1+4x^2}}\,dx$

36. Find $\dfrac{d}{dx}\displaystyle\int \ln x\,dx$.

In Problems 37 and 38, solve each differential equation using the given boundary condition.

37. $\dfrac{dy}{dx} = 3xy$; $y = 4$ when $x = 0$

38. $\cos y\dfrac{dy}{dx} = \dfrac{\sin y}{x}$; $y = \dfrac{\pi}{3}$ when $x = -1$

In Problems 39–51, find each integral.

39. $\displaystyle\int \frac{y\,dy}{(y-2)^3}$

40. $\displaystyle\int \frac{x}{(2-3x)^3}\,dx$

41. $\displaystyle\int \sqrt{\frac{1+x}{x^5}}\,dx$

42. $\displaystyle\int_{\pi^2/4}^{4\pi^2} \frac{1}{\sqrt{x}} \sin\sqrt{x}\,dx$

43. $\displaystyle\int_{1}^{2} \frac{1}{t^4}\left(1-\frac{1}{t^3}\right)^3\,dt$

44. $\displaystyle\int \frac{e^x+1}{e^x-1}dx$

45. $\displaystyle\int \frac{dx}{\sqrt{x}\,(1-2\sqrt{x})}$

46. $\displaystyle\int_{1/5}^{3} \frac{\ln(5x)}{x}\,dx$

47. $\displaystyle\int_{-1}^{1} \frac{5^{-x}}{2^x}dx$

48. $\displaystyle\int e^{x+e^x}\,dx$

49. $\displaystyle\int_{0}^{1} \frac{x\,dx}{\sqrt{2-x^4}}$

50. $\displaystyle\int_{4}^{5} \frac{dx}{x\sqrt{x^2-9}}$

51. $\displaystyle\int \sqrt[3]{x^3+3\cos x}(x^2-\sin x)\,dx$

52. Find $f''(x)$ if $f(x) = \displaystyle\int_0^x \sqrt{1-t^2}dt$.

53. Suppose that $F(x) = \displaystyle\int_0^x \sqrt{t}\,dt$ and $G(x) = \displaystyle\int_1^x \sqrt{t}\,dt$. Explain why $F(x) - G(x)$ is constant. Find the constant.

54. If $\displaystyle\int_0^2 f(x+2)\,dx = 3$, find $\displaystyle\int_2^4 f(x)\,dx$.

55. If $\displaystyle\int_1^2 f(x-c)\,dx = 5$, where c is a constant, find $\displaystyle\int_{1-c}^{2-c} f(x)\,dx$.

56. Area Find the area under the graph of $y = \cosh x$ from $x = 0$ to $x = 2$.

57. Water Supply A sluice gate of a dam is opened and water is released from the reservoir at a rate of $r(t) = 100 + \sqrt{t}$ gallons per minute, where t measures the time in minutes since the gate has been opened. If the gate is opened at 7 a.m. and is left open until 9:24 a.m., how much water is released?

58. Forensic Science A body was found in a meat locker whose ambient temperature is 10 °C. When the person was alive, his body temperature was 37 °C and now it is 25 °C. Suppose the rate of change of the temperature $u = u(t)$ of the body with respect time t in hour (h) is proportional to $u(t) - T$, where T is the ambient temperature and the constant of proportionality is -0.294.

(a) Write a differential equation that models the temperature $u = u(t)$ of the body at time t.

(b) Find the general solution of the differential equation.

(c) Find the particular solution of the differential equation, using the initial condition that at the time of death, $u(0) = 37$ °C.

(d) If the body was found at 1a.m., when was the murder committed?

(e) Will the body ever cool to 10 °C? Explain.

59. Radioactive Decay The amount A of the radioactive element radium in a sample decays at a rate proportional to the amount of radium present. Given the half-life of radium is 1690 years:

(a) Write a differential equation that models the amount A of radium present at time t.

(b) Find the general solution of the differential equation.

(c) Find the particular solution of the differential equation with the initial condition $A(0) = 10$ g.

(d) How much radium will be present in the sample at $t = 300$ years?

60. National Population Growth Barring disasters (human-made or natural), the population P of humans grows at a rate proportional to its current size. According to the U.N. World Population studies, from 2005 to 2010 the population of China grew at an annual rate of 0.510% per year.

(a) Write a differential equation that models the growth rate of the population.

(b) Find the general solution of the differential equation.

(c) Find the particular solution of the differential equation if in 2010 ($t = 0$), the population of China was 1.341335×10^9.

(d) If the rate of growth continues to follow this model, when will the projected population of China reach 2 billion persons?

Source: *U.N. World Population Prospects*, 2010 update.

CHAPTER 5 PROJECT Managing the Klamath River

There is a gauge on the Klamath River, just downstream from the dam at Keno, Oregon. The U.S. Geological Survey has posted flow rates for this gauge every month since 1930. The averages of these monthly measurements since 1930 are given in Table 2. Notice that the data in Table 2 measure the rate of change of the volume V in cubic feet of water each second over one year; that is, the table gives $\frac{dV}{dt} = V'(t)$ in cubic feet per second, where t is in months.

TABLE 2

Month	Flow Rate (ft^3/s)
January (1)	1911.79
February (2)	2045.40
March (3)	2431.73
April (4)	2154.14
May (5)	1592.73
June (6)	945.17
July (7)	669.46
August (8)	851.97
September (9)	1107.30
October (10)	1325.12
November (11)	1551.70
December (12)	1766.33

Source: USGS Surface-Water Monthly Statistics, available at http://waterdata.usgs.gov/or/nwis/monthly.

1. Find the factor that will convert the data in Table 2 from seconds to days. [*Hint:* 1 day = 1 d = 24 hours = 24 h(60 min/h) = (24)(60 min) (60 s/min).]

If we assume February has 28.25 days, to account for a leap year, then 1 year = 365.25 days. If $V'(t)$ is the rate of flow of water, in cubic feet per day, the total flow of water over 1 year is given by

$$V = V(t) = \int_0^{365.25} V'(t)dt.$$

2. Approximate the total annual flow using a Riemann sum. (*Hint:* Use $\Delta t_1 = 31$, $\Delta t_2 = 28.25$, etc.)

3. The solution to Problem 2 finds the sum of 12 rectangles whose widths are Δt_i, $1 \le i \le 12$, and whose heights are the flow rate for the ith month. Using the horizontal axis for time and the vertical axis for flow rate in ft^3/day, plot the points of Table 2 as follows: (January 1, flow rate for January), (February 1, flow rate for February), . . . ,(December 1, flow rate for December) and add the point (December 31, flow rate for January). Beginning with the point at January 1, connect each consecutive pair of points with a line segment, creating 12 trapezoids whose bases are Δt_i $1 \le i \le 12$. Approximate the total annual flow $V = V(t) = \int_0^{365.25} V'(t)\,dt$ by summing the areas of these trapezoids.

4. Using the horizontal axis for time and the vertical axis for flow rate in ft^3/day plot the points of Table 2 as follows: (January 31, flow rate for January), (February 28, flow rate for February), . . . , (December 31, flow rate for December). Then add the point (January 1, flow rate for December) to the left of (January 31, flow rate for January). Connect consecutive points with a line segment, creating 12 trapezoids whose bases are Δt_i, $1 \le i \le 12$. Approximate the total annual flow $V = V(t) = \int_0^{365.25} V'(t)\,dt$ by summing the areas of these trapezoids.

5. Why did we add the extra point in Problems 3 and 4? How do you justify the choice?

6. Compare the three approximations. Discuss which might be the most accurate.

7. Consult Chapter 7 and read about Simpson's Rule (p. xx). Can you see a way to use it to approximate the total annual flow?

8. Another way to approximate $V = V(t) = \int_0^{365.25} V'(t)dt$ is to fit a polynomial function to the data. We could find a polynomial of degree 11 that passes through every point of the data, but a polynomial of degree 6 is sufficient to capture the essence of the behavior. The polynomial function f of degree 6 is

$$f(t) = 2.2434817 \times 10^{-10}t^6 - 2.5288956 \times 10^{-7}t^5 + 0.00010598313t^4 - 0.019872628t^3 + 1.557403t^2 - 39.387734t + 2216.2455$$

Find the total annual flow using $f = f(t)$.

9. Use technology to graph the polynomial function f over the closed interval [0, 12]. How well does the graph fit the data?

10. A manager could approximate the rate of flow of the river for every minute of every day using the function

$$g(t) = 1529.403 + 510.330 \sin\frac{2\pi t}{365.25} + 489.377 \cos\frac{2\pi t}{365.25} - 47.049 \sin\frac{4\pi t}{365.25} - 249.059 \cos\frac{4\pi t}{365.25}$$

where t represents the day of the year in the interval [0,365.25]. The function g represents the best fit to the data that has the form of a sum of trigonometric functions with the period 1 year. We could fit the data perfectly using more terms, but the improvement in results would not be worth the extra work in handling that approximation. Use g to approximate the total annual flow.

11. Use technology to graph the function g over the closed interval [0, 12]. How well does the graph fit the data?

12. Compare the five approximations to the annual flow of the river. Discuss the advantages and disadvantages of using one over another. What method would you recommend to measure the annual flow of the Klamath River?