

Reasoning About Algebra and Change

Your experience with algebra in school likely was in one or two courses devoted to the subject. So you may be surprised to learn that algebra is increasingly recognized as a continuing topic throughout the curriculum, with ideas from algebra occurring as early as the first grade and with an increased use of symbols and equations in the K–6 mathematics curriculum, often under the guise of *algebraic reasoning*.

Algebra has acquired a somewhat negative image, and many cartoons play on the belief that algebra is difficult to understand and not too useful. This conception is far from the truth. When algebra is taught with sense-making always a goal, its power in solving many types of problems becomes apparent. Of course, this approach to teaching algebra is easier when the teacher has the background to help students make sense of the subject, including algebra topics now found in the elementary grades' curriculum.

The goal of the following chapters is to provide you with the ability and confidence to teach the ideas and skills that underlie algebra. This is not an algebra course devoted to building skills by manipulating numbers and variables. We do in fact assume that you have been exposed to such skills in earlier algebra classes, but, nonetheless, we have built in some review.

Chapter 12 in this part of *Reconceptualizing Mathematics* first centers on thinking about algebra in elementary school and what algebraic thinking is for elementary school children. This focus is integrated with a discussion of different facets of algebra, which lays the groundwork for the remaining chapters in Part II. We continue the discussion of quantitative reasoning as a way of understanding problem situations and our discussion of problem-solving heuristics. Chapter 13 focuses on using graphs to study quantitative relationships. Chapter 14 naturally follows this work by describing mathematical change and emphasizing what graphs tell us. Chapter 15 uses algebra to approach additional types of problems.

In each chapter we take opportunities to help you better understand the Common Core State Standards (CCSS) Standards for Mathematical Practice (SMP). The SMP are process standards that describe the ways students are expected to engage with mathematics. You can develop a better understanding as you reflect on the ways in which you engage in the mathematical practices as you work through the activities in Part II.

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What Is Algebra?

Chapter



There are many ways of thinking about algebra—as a symbolic language, as a study of patterns and functions, as generalized arithmetic, as reasoning about quantities, and as a powerful tool for solving problems.¹ This chapter introduces these different ways of thinking about algebra and provides a foundation for the next three chapters. We begin by thinking about algebra in elementary school. What is algebraic thinking for elementary school children?

12.1 Algebraic Reasoning in Elementary School

Long before children are introduced to formal algebra, they can engage in a level of generality that can be considered algebraic. Early algebra involves using symbols to express relationships, identifying patterns and making and expressing generalizations. We will discuss what algebra looks like in elementary grades in the first section, and then in subsequent sections, we will go deeper and revisit algebraic topics more formally.

Algebra provides a symbolic language that can be used to represent quantitative relationships, such as those in story problems. We use symbols in algebra in three primary ways: to stand for a particular value in an expression or sentence, to state formulas, and to express arithmetic and algebraic properties. The symbols most often used are letters of the alphabet, but in the primary grades other symbols, such as boxes, are sometimes used.

EXAMPLE 1

Jaime has 16 baseball cards but lost 4 of them. How many does he have left? We can represent this problem as $16 - 4 = \square$. Although you can substitute different numbers for the box, only one number will make this a true statement. ●

Elementary teachers should be concerned with the concept of equality as soon as students start writing symbols for number operations. Because elementary children usually encounter number sentences with the structure in Example 1, they often see the equals sign as a prompt to do the operation that precedes it. When elementary students solve $3 + 7 = \square + 2$, most put 10 or 12 in the box. They do not generally see the equal sign as “same as” or relational, but rather as an instruction to carry out the preceding procedure. They see the equal sign as separating the problem from the answer. Open number sentences, which include unknown starting points and unknown change problems, can encourage students to reason flexibly. We read in Chapter 10 about children as young as first and second grade who had a good understanding of equality, easily able to reason when shown equations with signed numbers and open boxes.

Notes

The Instructor Resource Manual contains several items of interest for you to read before beginning this chapter. First, information about Chapters 12, 13, 14, and 15 can be found in the Overview of Part II: Reasoning About Algebra and Change. You may find this information helpful in planning your course. Second, if you have not already done so, please read the Introduction to the Reconceptualizing Mathematics Program in the manual. Third, INSTRUCTOR NOTE 12 provides an overview of this chapter. Additional INSTRUCTOR NOTES in this manual are referenced throughout these chapters.

The first major topic of this section is that algebra is a symbolic language used to represent quantitative relationships.

Notes

Many prospective elementary teachers have negative feelings about the algebra course(s) they took. If they experienced algebra primarily as symbol manipulations, having such feelings is not surprising. We suggest that you explore their experiences by asking students a question such as “What do you remember about algebra?” Provide them with some hope that algebra can make sense and that they surely would not want their own students to have trouble learning algebra.

Discussions grounded in relationships provide another context for using symbols to represent quantitative relationships. In research, children were shown two boxes and were told that each box contained the same number of candies. Though the number of candies in each box was unknown, when three more candies were placed in the second box, children as young as 8 or 9 could express the number of candies in the boxes as $?$ and $? + 3$, and easily adopted N and $N + 3$ for the number of candies in the boxes.² In other words, they could conceptualize symbols as standing in for a number of values.

EXAMPLE 2

Karen is running a 5-mile race. If we let x represent the number of miles she has run at any time during the race, we can represent the number of miles remaining as $5 - x$. In this case, x can take on a variety of values between and including 0 and 5, depending on the distance Karen has run at any point in time. ●

These discussions of relationships and equality can lead children to create mathematical expressions and equations that use symbols to express relationships in preparation for more formal algebra. For example, consider the earlier example of candies in boxes. Children could reason about who has more. They understood that the child who had the second box had three more than the child who had the first box and could write an equation: $M = J + 3$. Children can then eventually write more complex algebraic equations.

EXAMPLE 3

Suppose a child (or child’s parent) pays \$3.00 for each superhero figurine and \$0.50 for each Superhero sticker. We can use symbols to tell us how much money was paid for figurines and stickers: $3f + 0.5s$, where f represents the number of figurines purchased and s represents the number of stickers purchased. ●

In these three examples, symbols are used to represent the values of certain quantities. These symbols are called *variables*.

A **variable** is a symbol used to stand for a value from a particular set of values.

As we have seen, children do not always begin by using conventional algebraic symbols. They can be encouraged to represent and reason about mathematics in their own way. Many people believe children benefit from opportunities that begin with their own intuition, and then they gradually adopt more formal representations and terminology.

Children also benefit from the study of patterns in numbers, shapes or objects, and even systems. Exercises in recognizing, describing, and extending patterns occur early in the curriculum. The primary focus of the next section (12.2) will be numerical patterns. However, you should be aware that repeating patterns of blocks or shapes, as in the following diagram, can appear in first-grade textbooks.



Children should be encouraged to notice and describe patterns, structure, and regularity in arithmetic operations. Number properties are the foundation of algebra. As you will see, these properties are used to evaluate and simplify mathematical expressions. Number properties (e.g., commutative property of multiplication or associative property of addition) must be understood and used in mental computation, though you may not be aware you are using them.

Similarly, children have some understanding of these properties, though they do not yet know their formal names. Consider this fourth-grader's logic for mentally obtaining the product of 4×20 : "Four times twenty is double four times ten because twenty is double ten."³ We can use symbols and properties to record this child's thinking. The property states that, in general, if we switch the order of the numbers being multiplied, the product is unaffected.

$$4 \times 20 = 2 \times (4 \times 10) \quad \text{because } 20 = 2 \times 10$$

$$4 \times (2 \times 10) = 2 \times (4 \times 10) \quad \text{associativity and commutativity of multiplication}$$

Children sometimes understand and state their observations more generally, as in the following explanation: "When you add zero to any number, you get the number you started with." This assertion is a partial statement of zero as an additive identity. It can be recorded with variables: $a + 0 = a$, where a is any number.

These properties are so important in both arithmetic and algebra that we restate them at the end of this section for future reference.



ACTIVITY 1 Listen Carefully Now

Here are some children's mental and written computations. If the child is using correct logic, discuss what properties he or she may be recognizing and describing. If the child is making an error, describe what property he or she appears to not understand.

- $45 + 15$ is the same as $50 + 10$ because I borrow 5 from the 15 to get to 50 and that leaves 10 more to add.
- 5×6 is 26 because 5×5 is 25 and 6 is one more than 5. Twenty-six is 1 more than 25.
- $\frac{2}{3} = \frac{8}{12}$ because $2 \times 4 = 8$ and $3 \times 4 = 12$. I have to multiply by the same thing in the numerator as the denominator.
- 23×9 is 23×10 minus 23.
- $\square + \square = 5$. I think the answer must be 8 because I add 3 to get to 0 and 5 more. That's 8. ■



ACTIVITY 2 Finding Properties

Tell what property or properties have been used:

- $a + (n + 49)$ is rewritten as $(a + n) + 49$.
- $(2b)c = 2(bc)$ (or $2bc$)
- $(29 \times \frac{7}{8} \times \frac{8}{7}) = 29 \times (\frac{7}{8} \times \frac{8}{7}) = 29 \times 1 = 29$
- $\frac{1}{2} \cdot (6 + \frac{4}{5})$ can be calculated by $\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot \frac{4}{5}$.
- $xy^2 + 0 = xy^2$
- $1 \cdot x = x$ (Note: x is commonly used in place of $1 \cdot x$.)
- $\frac{6^4}{5} \times \frac{1}{2} = \frac{1}{2} \times \frac{6^4}{5}$
- $z + -z = 0$

Use the properties to change the following expressions to make them easy to do mentally, and tell what properties you used:

- $(7 + 40) + 3$
- $25 \times (4 \times 72.7)$
- $24 \times 38 + 24 \times 12$ ■

Some representations (e.g., rectangular area, number lines, unit cubes) can help us see the structure and regularities of the properties. The distributive property of multiplication over addition can be represented with a rectangular area model; the area model is sometimes coupled with a number line. Commutativity and associativity of addition can be modeled on a number line or with unit cubes. The model is used to show that the result of the operations is the same.

Notes

SEE INSTRUCTOR NOTE 12.1A for information on student expectations on the topic of properties of arithmetic operations. The Common Core State Standards emphasize students' understanding and use of mathematical properties.

ACTIVITY 1

Answers

- Associative property of addition
- The child does not understand that multiplication is different from addition in that "1 more" means one more group of 5: $5(5 + 1) = 25 + 5$.
- 1 is the multiplicative identity, but the child may also be parroting a rule.
- Distributive property
- Additive inverses

ACTIVITY 2

Answers

- Associativity of addition
- Commutativity of multiplication
- Associativity of multiplication
- Additive inverses
- First, associativity of multiplication; then multiplicative inverses; and finally, 1 is the identity for multiplication.
- Distributivity of multiplication over addition
- Zero is the identity for addition.
- Identity property for multiplication
- $(40 + 7) + 3 = 40 + (7 + 3) = 50$; commutativity then associativity of addition
- $(25 \times 4) \times 72.7 = 100 \times 72.7 = 7270$; associativity of multiplication
- $24(38 + 12) = 24 \times 50 = 24 \times \frac{100}{2} = 12 \times 100 = 1200$; distributive property

Notes

MathClips



▶▶ I Expressing Regularity in Reasoning

THINK ABOUT ...

Answer

Construct an array or area model and show that the model for the commuted numbers is the same rectangle rotated.

SEE INSTRUCTOR NOTE 12.1B about using properties to compute mentally.

ACTIVITY 4

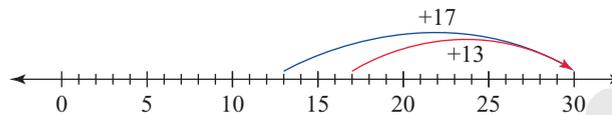
There are 41 toothpicks in Shape 10.

EXAMPLE 4

$$5 \times (10 + 7) = 5 \times 10 + 5 \times 7$$

**EXAMPLE 5**

$$13 + 17 = 17 + 13$$

**THINK ABOUT ...**

How can we model the commutative property of multiplication?

These same properties justify some of our routine algebraic manipulations, which helps make sense of these manipulations.

EXAMPLE 6

We can replace $5x + 3x$ with $8x$ because of the distributive property: $5x + 3x = (5 + 3)x$.

**ACTIVITY 3** To Help Explain

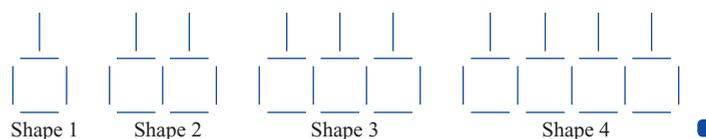
1. Use properties to explain why $5d - d \neq 5$.
2. Use a model or properties to explain why $3a + 2 \neq 5a$.

As you can see, children in elementary school have very powerful ways of reasoning when they are encouraged to describe patterns and regularity in arithmetic operations. Early algebra also includes an emphasis on teaching children how to make generalizations about “growing patterns.” Often this begins by asking them to describe how geometric patterns change. Children can be invited to describe how to draw more members of the same sequence. They use that context to anticipate or predict how many segments or corners will be in a particular picture in the sequence.

This context provides a space for thinking about variables, recognizing numerical patterns, describing relationships among variables, and linking multiple representations.

**ACTIVITY 4** Leah Made a Pattern

Look at the following sequence of shapes. Build Shape 5. How many toothpicks would be in Shape 10? Describe how you thought about building the shapes.



Children initially express the pattern iteratively; that is, they describe how many toothpicks are added to Shape 1 to get Shape 2 and how many are added to Shape 2 to get Shape 3, and so forth. It is far more complex to be able to say how many toothpicks are used to generate a particular shape without information about how many toothpicks form the shape before it in the sequence. When solving this problem, did you think about what changes and what is constant? Did you reflect on whether the change was constant or differed as the pattern grew? This involves powerful algebraic reasoning.

There is also mathematical richness in saying how the pattern grows. If children see the pattern growing in different ways, they can describe the growth with different rules. A child might say that four toothpicks are added to Shape 1 to get Shape 2 and four more to get Shape 3, focusing on the four toothpicks added to the right of the figure each time. Alternatively, a child might say that they see another figure being added, with overlapping toothpicks “erased” (i.e., Shape 2 is two shapes of five toothpicks minus the one toothpick in the middle). Rich conversations can ensue in thinking about what is the same about the representations: $5 + 4$ and $(2 \times 5) - 1$. In describing growing patterns, children learn how to make generalizations and express correspondence. The geometric arrangement of the objects in the pattern can help them bridge to generalizing for n objects and writing algebraic expressions. This is an early introduction to function. We will learn more about functions in later sections.

DISCUSSION 1 Algebra in Some Elementary Classrooms

Can you solve the following two problems? Are you surprised that these types of problems are being solved in elementary classrooms?

- How many hexagonal tables would it take to seat 25 guests? ◀

Assume that 6 guests could be seated at one hexagonal table, and if there are two or more hexagonal tables, they are arranged as shown in the diagram. Guests can be seated at each side of a hexagonal table except where they are joined.



- What are some mathematically complex ways to arrive at 9 (e.g., $350 - 341$)? ◀

Read this excerpt from a report on mathematics in classrooms in the Lebanon, Oregon, school district, published December 29, 2008, in the Portland newspaper *The Oregonian*.

LEBANON—Lori Haley and Mya Corbett hunch over a pile of yellow hexagons, trying to figure out how many hexagonal tables it would take to seat 25 guests. The pair want to get the answer, but what they’re really itching to do is to come up with a formula that will tell them how many people they could seat for any given number of tables.

Suddenly, the girls detect a pattern, and one shouts: “ $(t \times 4) + 2 = s!$ ” They try it on one table, two tables, eight tables—it works. They beam, flashing smiles. . . . Lori and Mya just started third grade.

Visit a Lebanon elementary math class, and you will see:

- First-graders set up and solve formulas such as $9 - x = 5$, as they did when Raylene Sell talked with her class about “some teddy bears” walking away from the classroom rug, leaving five behind.
- Third-graders suggest mathematically complex ways to arrive at 9: $-219 + 228$ or $(10 \times 5) - 40 - 1$, or even $(3 \times 3) + (8 \times 8) - ((4 \times 4) + (4 \times 4)) - 32$. ■

One researcher observed that arithmetic is not about obtaining results, but rather it is about learning the means by which one obtains particular solutions. Similarly, algebra is not about the manipulation of symbols for particular results but about expressing generality about patterns, the properties of numbers, and the rules of arithmetic. What is algebraic

DISCUSSION 1

The yellow hexagons used in this classroom were from a set of pattern blocks. Here, it appears that the tables are set up so that every two tables are linked along one edge of each; thus, one table would seat 6 people, two tables would seat 10, three tables would seat 14, etc.

Make sure students note the grade levels here. This discussion should help promote the earlier claim that algebraic problems are now appearing in elementary school classrooms.

Notes

Many students will (mistakenly) pronounce commutativity as “communitativity.” As future teachers, they should be able to pronounce and spell all these property names.

Another major topic of this section is the review and consolidation of these properties. A review of properties of operations is also provided in Appendix F: A Review of Some Rules. You may need to clarify for students that the conventional “any choice” means “every choice.” Closure properties for addition and multiplication are not included here because we do not explicitly use them in the Part II text.

thinking for elementary children? We need to encourage children to focus on relationships and to describe patterns and structure; in short, we need to support them to “see the general in the particular.”

Properties

Let a , b , and c be any choice of numbers.

Properties of addition

1. Commutativity of addition: $a + b = b + a$. For example, $3x + 5x = 5x + 3x$.
2. Associativity of addition: $a + (b + c) = (a + b) + c$. For example, $362d + (8d + 27) = (362d + 8d) + 27$.
3. Zero is the identity for addition: $0 + a = a$ and $a + 0 = a$. For example, $0 + 505 = 505$ and $9t + 0 = 9t$.
4. Existence of additive inverses: For each number a , there is another number, denoted $-a$, such that $a + (-a) = 0$ and $-a + a = 0$. Each of a and $-a$ is the additive inverse of the other. For example, $7x$ and $-7x$ are additive inverses of each other, because $7x + (-7x) = 0$.

Properties of multiplication

5. Commutativity of multiplication: $a \times b = b \times a$. For example, $2r \times 3t = 3t \times 2r$.
6. Associativity of multiplication: $a \times (b \times c) = (a \times b) \times c$. For example, $4z \times (25z \times 3) = (4z \times 25z) \times 3$.
7. One (1) is the identity for multiplication: $1 \times a = a$ and $a \times 1 = a$ for any number a . For example, $1 \times \frac{x}{7} = \frac{x}{7}$ and $3 \times 1 = 3$.
8. Existence of multiplicative inverses: For each nonzero number a , there is another number b , such that $a \times b = 1$ and $b \times a = 1$. For example, $\frac{3}{5}$ and $\frac{5}{3}$ are multiplicative inverses of each other because $\frac{3}{5} \times \frac{5}{3} = 1$. We also say that $\frac{3}{5}$ and $\frac{5}{3}$ are reciprocals, or multiplicative inverses, of each other. In the same manner, we can say k and $\frac{1}{k}$ are reciprocals or multiplicative inverses of each other.

A property involving both addition and multiplication

9. Distributivity of multiplication over addition (often referred to simply as the distributive property): $a \times (b + c) = (a \times b) + (a \times c)$ for any numbers a , b , and c . The form $(b + c) \times a = (b \times a) + (c \times a)$ is also useful. For example, $6 \times (3a + 2) = (6 \times 3a) + (6 \times 2)$ and $(4.2 + 7) \times r = 4.2r + 7r$.

The \times sign is easily confused with the variable x , so both the raised dot (as in $a \cdot b$ or $2 \cdot b$) and juxtaposition (as in ab or $2b$) are commonly used in algebra to indicate multiplication when doing so is not confusing. We also do not need a multiplication symbol when parentheses are used, i.e., $6 \times (4a + 3)$ could be written simply as $6(4a + 3)$.

Commutativity of multiplication is then compactly expressed by the statement $ab = ba$ for every choice of numbers or algebraic expressions a and b . Any algebraic expression can play the roles of a and b . For example, commutativity of multiplication assures that $(3x + 8)(2x + 5)$ and $(2x + 5)(3x + 8)$ will be equal for any choice of the variable x . Other properties involving multiplication can be restated similarly.

TAKE-AWAY MESSAGE . . . Algebra provides a symbolic language that can be used to represent quantitative relationships; that is, word problems can be solved by first expressing them as equations. Early algebra is not formal algebra, but instead it has a focus on generalizing and expressing relationships.

A variable is a symbol used to stand for a value from a particular set of values. Variables can be used (1) in expressions such as $5 - x$, (2) in formulas such as $A = bh$, and (3) to describe properties such as $a + b = b + a$ for any values a and b .

Learning Exercises for Section 12.1

- Evaluate and then express each of the following statements as a general property, using variables. Give the name of the property.
 - $(18 \times 93) + (18 \times 7)$ can be calculated mentally and exactly by $18 \times (93 + 7)$.
 - 12 nickels plus 8 nickels has the same penny value as 20 nickels.
 - $(231 + 198) + 2$ can be calculated exactly by $231 + (198 + 2)$.
 - $(17 \times 25) \times 4$ can be calculated mentally and exactly by $17 \times (25 \times 4)$.
 - $\frac{117}{298} \times \frac{39}{39}$ is an easy mental calculation.
 - $\frac{0}{24} + \frac{13}{35}$ is an easy mental calculation.
 - Each of 4 pockets has a dime and 7 pennies. Calculate the total value in two ways.
- Consider the related problems in parts (a–c) that lead up to the symbolic equation $a + d - d = a$.
 - Is $65 + 108 - 108$ equal to, larger than, or smaller than 65?
 - Is $65 + 4\frac{1}{4} - 4\frac{1}{4}$ equal to, larger than, or smaller than 65?
 - Is $65 + d - d$ equal to, larger than, or smaller than 65, or is it not possible to tell?
 - Write a set of sentences that leads up to the sentence “ $g - h + h = g$.”
- Consider each statement below. Replace the symbol ∇ with the relation symbol $=$, $<$, or $>$ in turn, and say which replacements give true sentences. The first one is done for you.
Example: $x \nabla x$ is true if $=$ is substituted for ∇ (i.e., $x = x$), but not true if either $<$ or $>$ is substituted for ∇ .
 - If $x \nabla y$, then $y \nabla x$.
 - If $x \nabla y$ and $y \nabla z$, then $x \nabla z$.
- The following problems are typical of those asked in grade school. Express each one using an equation with one variable:
 - Malia has \$2.30 but wants to buy a movie ticket, which costs \$4.00. How much more does she need?
 - Leesha had 8 stuffed animals and received 3 more. How many does she have now?
 - Jeremy downloaded 6 songs from iTunes onto his iPod. He already had 360 songs on his iPod. How many songs does he have on his iPod now?
 - Regis is a teacher. He bought 10 boxes of colored markers for his students. Each box has 12 markers. He packages them into bags for each of his 30 students. If each student gets the same number of markers, how many markers are in each bag?
- Choose a number, and keep it secret from others. Add 8 to your number, then multiply by 4, subtract 3, add 7, divide by 4, and subtract 9. You should now have the number you started with. Why does this work?
 - Make up a similar problem with at least five steps.
- Four people have these amounts of money: x , $2x$, $x + 12$, and $x - 3$.
 - Write an algebraic expression that gives the total amount of money they have.
 - Write an expression that gives each person’s share if they share the money equally.
 - Write an expression that gives the total remaining if each person spends \$15.
 - If the four people originally had a total of \$119, how much did each person have?
 - Describe a situation that leads to the equation $5x + 9 = \$36$, and then solve the equation and use it to explain the situation.
- Joan is x years old now. Write an algebraic expression for each part.
 - Joan’s age in 10 years
 - Joan’s age in y years
 - Joan’s age 3 years ago
 - Joan’s age n years ago
 - Joan’s age when she is twice as old
- Kay has k songs on her iPod. Write an algebraic expression for each situation described.
 - Lenore has 12 fewer tunes on her iPod than Kay has.
 - Kay downloads n new tunes for her iPod.

Notes

Learning Exercises 12.1:

Students do not have answers for Learning Exercises 1(c–g), 2–5, 8, 9, and 11–14. Note that students have answers to all the Supplementary Learning Exercises for all sections in the Student Solutions Manual. Be sure to assign Learning Exercises 3, 5, 12, and 13.

Notes

- c. On her iPod, Minnie has a third as many tunes as Kay has after part (b).
 d. Nan has twice as many tunes as Lenore has [from part (a)].
9. An item is priced at d dollars. Write an algebraic expression for each situation described.
- the cost of six of the items
 - the cost of n of the items
 - the cost of two of the items, along with \$17.95 worth of other items
 - the reduced price if the item is sold at 30% off
 - your change if you pay for the item with two \$20 bills
 - the amount of tax on the item if the sales-tax rate is 7%
10. A researcher posed each of the following problems to Brazilian street children.⁴ They could solve one but not the other. Why do you suppose this happened? Which one could they solve?
- ▶ A boy wants to buy chocolates. Each chocolate costs 50 cents. He wants to buy 3 chocolates. How much money does he need? ◀
 - ▶ A boy wants to buy chocolates. Each chocolate costs 3 cents. He wants to buy 50 chocolates. How much money does he need? ◀
11. Use an array, a rectangular area, a number line, or base-ten blocks to model each of these properties for a particular instance.
- associative property of multiplication: $(3 \times 5) \times 2 = 3 \times (5 \times 2)$
 - associative property of addition: $1.7 + 3.3 = 2.0 + 3.0$ because $1.7 + (0.3 + 3.0) = (1.7 + 0.3) + 3.0$
 - existence of multiplicative inverses: $\frac{2}{3} \times \frac{3}{2} = 1$ and $\frac{3}{2} \times \frac{2}{3} = 1$
 - commutativity of multiplication (show on a number line *and* with a rectangular area model): $3 \times 5 = 5 \times 3$
12. Use an array, a rectangular area, a number line, or base-ten blocks to model each of these properties for a particular instance.
- distributive property of multiplication over addition: $7 \times (44 + 6) = 7 \times 44 + 7 \times 6$
 - associative property of addition: $96 + 21 = 96 + (4 + 17) = (96 + 4) + 17$
 - commutative property of multiplication: $17.4 \times 3.5 = 3.5 \times 17.4$
 - associative property of multiplication: $(4 \times 3) \times 6 = 4 \times (3 \times 6)$
 - commutative property of addition: $3\frac{1}{4} + 8\frac{3}{8} = 8\frac{3}{8} + 3\frac{1}{4}$
13. Use an array, a rectangular area, a number line, or base-ten blocks to model each of these properties for a particular instance.
- distributive property of multiplication over addition: $14 \times 6\frac{1}{2} = 14 \times (6 + \frac{1}{2}) = 14 \times 6 + 14 \times \frac{1}{2}$
 - associative property of addition: $-6 + 20 = -6 + (6 + 14) = (-6 + 6) + 14$
 - commutative property of multiplication: $8\frac{2}{3} \times 7\frac{1}{2} = 7\frac{1}{2} \times 8\frac{2}{3}$
 - associative property of multiplication: $(3 \times 27) = 3 \times (3 \times 9) = (3 \times 3) \times 9$
 - commutative property of addition: $-15 + -4 = -4 + -15$
14. Use models and properties to explain why $7x - x \neq 7$.
15. Following is a child's description of his or her computation of 21×15 .
- ▶ To find twenty-one times fifteen, I multiply 20×15 because that is easy. Two times fifteen is thirty, so for twenty I add a zero and get three hundred. But I need one more than twenty. Three hundred and one! ◀
- If the child is using correct logic, discuss what property he or she may be recognizing and describing. If the child is making an error, describe what property he or she appears not to understand.
 - Model the child's correct thinking or use the model to show the correct thinking.

Supplementary Learning Exercises for Section 12.1

1. Is each equation true or false? If true, which property is demonstrated?
 - a. $(x + 3) + 5(x + 3) = 6(x + 3)$
 - b. $32 \times 8 = 30 \times 8 + 2 \times 8 = 240 + 16 = 256$
 - c. $(2x + 17) + 13 = 2x + (17 + 13)$
 - d. $3x + 5x + 17 \times 3x = 3x + 5x + (17 \times 3) + (17x)$
 - e. $16 \times 25 = (4 \times 4) \times 25 = 4(4 \times 25) = 4 \times 100 = 400$
 - f. $16 \times 25 = 10 \times 25 + 6 \times 25 = 250 + 150 = 400$
 - g. $41 \times 28 = 40 \times 28 + 8$
 - h. $3x + 5x + 51x = 59x^3$
 - i. $2a + 2b = 2(a + b)$
 - j. $3x + 5x + 17 \times 3x = 3x + 5x + (17 \times 3)x$
 - k. $39x + 21y + 40z \times 0 = 0$
 - l. $(39x + 21y + 40z) \times 0 = 0$
 - m. $2k + 13v + 24z = 2k + 24z + 13v$
 - n. $\frac{24}{7} \times \frac{13}{13} = \frac{24}{7}$
2. For each statement below, let @ stand for =, <, or >. In each statement, which symbols are true for values $b, m, n, r, s,$ and t ? (For example, if < is substituted for @, then for any value c , we know that $c @ c$ is not true because it is not possible for $c < c$.)
 - a. $b @ b$
 - b. If $m @ n$, then $n @ m$.
 - c. If $r @ s$ and $s @ t$, then $r @ t$.
3. Which of these formulas are true? If they are false, why?
 - a. If n is the measure of each side of a pentagon (a five-sided polygon), then the perimeter P of the pentagon can be found by the formula $P = 5n$.
 - b. The area A of the same pentagon can be found by the formula $A = n^2$.
 - c. If the side of a square is $2s$, then the area of the square is $A = 4s^2$.
 - d. If a polygon has sides $f, g, h, k, m, n,$ and q , then the perimeter P of the polygon is $P = q + n + m + k + h + g + f$.
 - e. If the radius of a circle is 5 cm, then the circumference of the circle is 25 cm.
4. A group of friends collect comic books. Write an algebraic equation or inequality for each of the following statements in terms of the number c of comic books that Casey owns. You may introduce new variables (but make clear what they represent).
 - a. Kaye has 3 times as many comic books as Casey.
 - b. Nancy has more than twice as many comic books as Casey.
 - c. Jinfa has only half as many comic books as Casey.
 - d. Fortuna has 4 more comic books than Kaye has.
 - e. Melissa has 2 fewer comic books than Fortuna has.
5. Suppose n is the number of sweaters owned by Alicia, $n + 2$ is the number of sweaters owned by Kim, and $2n$ is the number of sweaters owned by Samme. Write an algebraic expression for each of the following statements. Simplify the expression if it is possible to do so.
 - a. The total number of sweaters they own altogether
 - b. Samme gives 6 of her sweaters to Goodwill, and Alicia receives 2 more sweaters for her birthday. How many sweaters do they now own altogether?
 - c. Later, they learn that Carolyn lost all her clothing when her home was destroyed by a hurricane. Each of the others gave her 2 sweaters. How many sweaters do the three now own in all?
6. Give the answer to each part using mental calculation. Write which property enabled you to do the calculation mentally.
 - a. $(59 \times 25) \times 4$
 - b. $(4 \times 3.98) + (6 \times 3.98)$
 - c. $\frac{82}{71} \times \frac{71}{82} \times \frac{97}{106}$
 - d. $(24 \times 1.50) + (24 \times 4.50)$

Notes

Answers for all Supplementary Learning Exercises are in the Student Solutions Manual.

Notes

If you have the time, you may want to supplement this section by looking at additional nonnumeric patterns. Again, depending on time, you may wish to delete work with the different types of patterns discussed here. It is sufficient to notice the pattern without necessarily naming different types.

The focus on numerical patterns leads more directly to algebraic expressions.

7. Sydney sold b boxes of Girl Scout cookies. Write an algebraic expression for each situation described.
 - a. Susan sold 7 boxes more than Sydney.
 - b. Taylor sold 2 more than the total number of boxes that Sydney and Susan sold.
 - c. Sydney sold b plus c more boxes of Girl Scout cookies.
 - d. Rachel sold 2 fewer boxes than twice as many boxes as Susan sold.
8. Begin with your age and follow these steps. The final result will be 25.
 - a. Add 14 to your age. Multiply the result by 2. Subtract 4. Divide by 2. Add 13. Subtract your age.
 - b. Why does this work? Modify the above steps so that the result will be your age.

12.2 Numerical Patterns and Algebra

Mathematics is sometimes described as the *science of patterns* because it involves the study of patterns in numbers, in shapes or objects, and even in systems. This section highlights the importance of patterns in mathematics. Exercises in recognizing, describing, and extending patterns occur in the curriculum even in the first grade. Numerical patterns often involve some calculation practice within the larger goal of seeing what the pattern is, so patterns can disguise a drill. In this section our focus is on patterns and how to represent them algebraically.

Following is an example of a typical numerical pattern exercise given in the early grades.

- What numbers go in the blanks to continue the pattern?

7, 17, 27, 37, ____, ____ ◀

In later grades, children might be asked to tell what the 51st number in the pattern is, a more challenging exercise, with the intent that the child *not* fill in all the intervening blanks.

THINK ABOUT ...

Jan decided that the 51st number would be 507. How do you think Jan reasoned?

The ability to predict is an important by-product of recognizing a pattern.

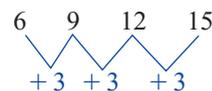
EXAMPLE 7

Predict the next four numbers in each sequence of numbers. (The entries are often called **terms**, especially if a variable is involved.) The dots (. . .) are used to indicate that the pattern continues forever.

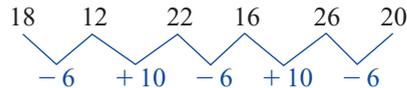
1. 6, 9, 12, 15, . . .
2. 18, 12, 22, 16, 26, 20, . . .
3. 20, 19.75, 19.5, 19.25, . . .
4. 2, 6, 12, 20, 30, . . .

Solution

1. The numbers seem to increase by 3 each time, so a good prediction is that the next four numbers would be 18, 21, 24, and 27.

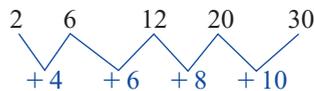


2. This pattern is more complicated.



But it looks as though $20 + 10$, or 30, would be the next number, and then $30 - 6 = 24$, $24 + 10 = 34$, and $34 - 6 = 28$ would follow.

3. Subtracting 0.25 (or adding -0.25) gets from one term to the next, so the next four numbers could be $19.25 + -0.25 = 19$, $19 + -0.25 = 18.75$, then 18.5, and 18.25—that is, the next four terms are 19, 18.75, 18.5, and 18.25.
4. This sequence of numbers also has a pattern.



It is reasonable to predict $30 + 12 = 42$, $42 + 14 = 56$, $56 + 16 = 72$, and $72 + 18 = 90$ to be the next four numbers: 42, 56, 72, and 90. ●

The type of pattern in the first and third sequences in Example 7 is so common that it has a name. As a teacher, you should know about it, even though the terms may not come up in elementary or middle school. A pattern can be formed by adding the same number, called the **common difference**, to any number in a list to get the next number. For example, in the first pattern in Example 7, the common difference is 3. In the third pattern, we could say that the common difference is -0.25 . A list of numbers in which the rate of change is constant is called an **arithmetic sequence**.

ACTIVITY 5 | Beg to Differ?

- Which, if any, of the following sequences of numbers are arithmetic sequences? Identify the common difference d for any arithmetic sequence you find, and give the next three numbers in the pattern.
 a. 18, 23, 28, 33, ... b. 7.2, 7.9, 8.6, 9.3, 10, ... c. 112, 109, 106, 103, 100, ...
- Write the first five numbers in an arithmetic sequence that starts with 18 and has a common difference of 4.
- Identify the 20th number in each of the arithmetic sequences in Problem 1, without putting in all the intervening numbers. (*Hint*: How many times must the common difference be added to the first number to get the 20th number?) ●

The kind of thinking that you probably did for Problem 3 in Activity 5 allows you to predict *any* number—even the 100th or the 2000th—in an arithmetic sequence, *without having to find the intervening terms*. To get the n th term, you must add to the first number in the sequence the common difference, $n - 1$ times:

$$\text{The } n\text{th number} = \text{the first number} + (n - 1) \times (\text{the common difference})$$

We can express this general idea more briefly by using algebraic symbols. Suppose that in an arithmetic sequence the first number is a and the common difference is d . Then the n th number in the arithmetic sequence is given by this equation:

$$\text{The } n\text{th number} = a + (n - 1)d$$

Use the formula to check that the 51st numbers in the arithmetic sequences in Activity 5, Problem 1 are 268, 42.2, and -38 , respectively.

Notes

We do not use the term *recursion* in this text because it does not come up specifically in grades K–8 textbooks.

As an adjective, *arithmetic* is pronounced “ar-ith-MET-ik.”

ACTIVITY 5

Answers

- All are arithmetic sequences.
 - $d = 5$; 38, 43, 48
 - $d = 0.7$; 10.7, 11.4, 12.1
 - $d = -3$; 97, 94, 91
- 18, 22, 26, 30, 34
- $18 + 19 \times 5 = 113$
 - $7.2 + 19 \times 0.7 = 20.5$
 - $112 + 19(-3) = 55$

Problem 3 is a key question. It gives a basis for the following expressions involving the n th number.

Notes

The barometric pressure drops with an increase in altitude; hence, water boils at a lower temperature. The data here are rule-of-thumb data.

ACTIVITY 6

Answers

- 209; 207; 4000; $212 - \frac{n}{500}$
- 192°F

3. When baking a cake, boiling water makes the leavening agents (e.g., baking powder, eggs) react more quickly, which causes the cake to rise too fast. A sunken, coarse cake is the result.

We do not usually emphasize geometric sequences because they are relatively advanced for the usual elementary curriculum. Nevertheless, they can be very useful to any course that includes prospective middle school teachers. You may prefer to skip over this content.

ACTIVITY 7

Answers

- 15, 45, 135, 405
- $5 \times 3^{100-1}$

 **Focus on SMP MP8**

Mathematically proficient students look for and express regularity in repeated reasoning. As you look for patterns in the differences, you are recognizing a similarity. This involves attention to specific numbers and details. As you formulate expressions for finding the n th number, you are expressing and using regularity.

 **THINK ABOUT ...**

Why is it $n - 1$ rather than n in the expression $a + (n - 1)d$ for finding the n th number in an arithmetic sequence?

 **ACTIVITY 6** Sally Moves to the Rockies

Water does not always boil at 212° Fahrenheit. The boiling point changes depending on the altitude.

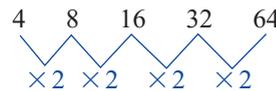
- Find a pattern in this table of altitudes in feet above sea level and temperatures in degrees Fahrenheit. Find the numbers that are missing. Then write an expression for the boiling point at an altitude of n feet.

Altitude above sea level (ft)	0	500	1000	1500	2000	2500	?	n
Boiling point of water (°F)	212	211	210	?	208	?	204	?

- Sally moves to the Rockies and now lives at 10,000 ft above sea level. What is the boiling point of water where Sally now lives?
- Why does the boiling point of water affect baking a cake? ■

Notice in Activity 6 that the ability to predict numbers in the boiling-point column from numbers in the altitude column is quite useful. With regular steps (500 ft here) in the first column, the decrease of 1°F at each step in the second column could be used together to predict the boiling point at other altitudes.

Here is another type of numerical pattern that has a name: A pattern in which each number in a list is obtained by *multiplying* the number that precedes it by the *same factor* is called a **geometric sequence**. For example, 4, 8, 16, 32, 64, . . . is a geometric sequence, with the factor being 2:



Geometric sequences usually get less attention than arithmetic sequences in the elementary school curriculum.

 **THINK ABOUT ...**

The factor, which is 2 in the previous sequence, is often called the **common ratio**. Do you see why?

 **ACTIVITY 7** Go Forth and Multiply

Consider a geometric sequence that starts with 5 and has a common ratio 3.

- Write the next four numbers in the geometric sequence.
- Write an expression that gives the 100th number in the sequence. [*Hint*: In part 1, how many times did you multiply by 3?]

3. The table in Activity 6 organized the data so that we could predict entries in the second column from entries in the first column. Based on the table below, give an algebraic expression for the n th number in a geometric sequence that starts with a and has common ratio r .

Which number in the sequence?	1st	2nd	3rd	4th	5th	...	n th
The term in that position	A	ar	ar^2	ar^3	ar^4	...	?

4. If the geometric sequence were 27, 9, 3, . . . , what is the common ratio r ? Does the algebraic expression in 3 apply here?

THINK ABOUT ...

Someone says, "The number of bacteria increases geometrically day by day." What does this mean?

Many sequences are neither arithmetic nor geometric. The sequence 2, 6, 12, 20, 30, . . . is neither an arithmetic nor a geometric sequence. Even though we can predict the next numbers in that sequence, it is not easy to see how we could predict, say, the 100th number without putting in all the in-between numbers. However, notice that the **first** number $2 = 1 \cdot 2$, the **second** number $6 = 2 \cdot 3$, the **third** number $12 = 3 \cdot 4$, the **fourth** number $20 = 4 \cdot 5$, and the **fifth** number $30 = 5 \cdot 6$. Do you see a pattern in the products that would enable you to see that the 99th number in the sequence would be 9900?

Sometimes a problem can be solved by looking at simpler or specific cases, organizing the data from those cases, and looking for a pattern in those data. Activity 8 illustrates these useful techniques.

ACTIVITY 8 How Do You Get Started?

Patterns can be useful in solving problems that have several specific cases, such as this one:

Figure out a shortcut for squaring any number ending in 5.

For example, with the shortcut you will be able to give the product 75^2 easily, without paper and pencil (or a calculator). Here is one way to approach a problem like this one. Let us look at some specific cases, possibly $25^2 = 625$, $65^2 = 4225$, $15^2 = 225$, $45^2 = 2025$, and $35^2 = 1225$. At this point, it looks as though each product ends in 25 (ending in 5 is not a surprise). But how could we predict the other digits? A table is a convenient way to organize numerical data. ■

EXAMPLE 8

Consider the fraction $\frac{5}{27} = 0.185185185 \dots$ (the digits 185 repeat forever). What digit will be in the 2000th decimal place in the decimal for $\frac{5}{27}$?

Solution

The block of digits repeats, so there is a pattern. But how can we predict the 2000th decimal place without putting in a lot of 185s? Here is one way to think: The repeating block is 3 digits long, so we need to find out how many full blocks of 3 digits will get us to the 2000th decimal, or close to it. A moment's thought might suggest that what we want to know is how many 3s there are in 2000. That thinking suggests division. Because $2000 \div 3 = 666 \text{ R } 2$, to get to the 2000th decimal place will take 666 full 1-8-5 blocks, with two additional digits from the next 1-8-5 block. Do you see that the digit in the 2000th decimal place must be 8? ■

Patterns will also be featured in the next section, where we relate patterns to what are called functions.

Notes

ACTIVITY 7

Answers

3. ar^{n-1}

Tables such as this one are used to introduce functions in Section 12.3.

4. $\frac{1}{3}$

The 2, 6, 12, 20, 30, . . . pattern will occur in the next activity, so it is useful to draw attention to this question.

ACTIVITY 8

Even with the numerical example lead-in, we usually arrive at two ideas: $n(n+1)$ and n^2+n , where n is the number of tens. Some students may not see that these expressions are algebraically equivalent. SEE INSTRUCTOR NOTE 12.2A for an algebraic justification.

Notes

Learning Exercises 12.2

Students do not have answers for Learning Exercises 1(a–h, k–l), 2–6, 8(b), 9, 12(c), 14, and 15. Be sure to assign Learning Exercises 9 and 10. Learning Exercise 12 is a good problem for a write-up.

Learning Exercises 2 and 3: These exercises appear here in case someone is not aware of these shortcuts.

Learning Exercises 4(c) and 4(d): The text discussion does not give a worked-out example for these exercises.

TAKE-AWAY MESSAGE . . . The study of patterns underlies a good part of mathematics. Many numerical patterns can be described in ways that enable us to determine the actual number at any location in the pattern. For example, the n th term in an arithmetic sequence starting with a and having common difference d is given by $a + (n - 1)d$. The n th term in a geometric sequence that starts with a and has common ratio r is given by ar^{n-1} . In general, an unfamiliar problem might be solvable by looking at specific cases, organizing the data, and looking for a pattern.

Learning Exercises for Section 12.2

- For each pattern, give the next four entries and any particular entry requested, as suggested by the pattern. Assume that the patterns continue indefinitely.
 - ABABAB ____ ; the 100th entry is ____.
 - ABBAABBA ____ ; the 63rd entry is ____.
 - 6.5, 7.3, 8.1, 8.9, ____, ____, ____, ____; the 20th entry is ____.
 - 100, 95, 90, 85, ____, ____, ____, ____; the 30th entry is ____.
 - 2, 6, 18, 54, ____, ____, ____, ____.
 - 5, 2.5, 1.25, 0.625, ____, ____, ____, ____.
 - $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$; the 100th entry is ____.
 - $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$; the 100th entry is ____.
 - 2, 4, 6, 8, 2, 4, 6, 8, ____, ____, ____, ____; the 30th entry is ____.
 - 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, 1, 4, ____, ____, ____, ____; the 40th entry is ____.
 - Which of the sequences in parts (a–j) are arithmetic sequences?
 - Are there any geometric sequences in parts (a–j)? If so, tell which parts.
- Examine these (correct) calculations for a pattern, and look for an easy rule for multiplying by a power of 10. Write the rule.

$12.3457 \times 10 = 123.457$	$23 \times 10 = 230$
$12.3457 \times 100 = 1234.57$	$23 \times 100 = 2300$
$12.3457 \times 1000 = 12,345.7$	$23 \times 1000 = 23,000$
$12.3457 \times 10,000 = 123,457$	$23 \times 10,000 = 230,000$
$12.3457 \times 100,000 = 1,234,570$	$23 \times 100,000 = 2,300,000$

 - Why does your rule work?
 - Examine these (correct) calculations for a pattern, and look for an easy rule for dividing by a power of 10. Write the rule.

$512.345 \div 10 = 51.2345$	$8.41 \div 10 = 0.841$
$512.345 \div 100 = 5.12345$	$8.41 \div 100 = 0.0841$
$512.345 \div 1000 = 0.512345$	$8.41 \div 1000 = 0.00841$
$512.345 \div 10,000 = 0.0512345$	$8.41 \div 10,000 = 0.000841$
$512.345 \div 100,000 = 0.00512345$	

 - Why does your rule work?
 - How can this idea be used to find 1% of a known amount? 10% of a known amount?
- Use the equation (n th term) = $a + (n - 1)d$ for an arithmetic sequence starting with a and having a common difference d to find each of the following numbers:
 - the 500th number in the sequence, 10, 14, 18, 22, 26, . . .
 - the 25th number in the sequence, $6\frac{3}{4}, 7\frac{1}{2}, 8\frac{1}{4}, 9, \dots$
 - the first number in an arithmetic sequence that has a common difference of 5 and 257 as its 51st term
 - the first number in an arithmetic sequence that has a common difference of 4.5 and 466 as its 101st term

5. The numbers 1, 1, 2, 3, 5, 8, . . . give an example of a **Fibonacci** (pronounced “fee-ba-NAH-chee”) **sequence**, which is a pattern that appears in nature, art, and geometry.

- a. What are the next four numbers in that Fibonacci sequence? (*Hint*: Look at two consecutive numbers in the sequence and then the next one.)
 b. Amazingly, the n th number in that Fibonacci sequence is

$$\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

Verify that for $n = 1$ and $n = 2$, the expression does give the first two numbers in the pattern, 1 and 1.

- c. Give decimals for the first ten ratios of consecutive Fibonacci numbers:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \dots$$

The ratios get closer and closer to a special value that is called the **golden ratio**. The golden ratio is often used in art to make pleasing proportions.

6. a. What is the n th fraction in the following sequence?

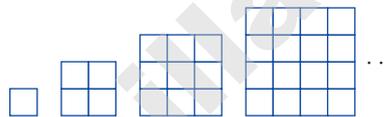
$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$$

- b. What is the *sum* of the first n of those fractions? To what number is the sum getting closer and closer?

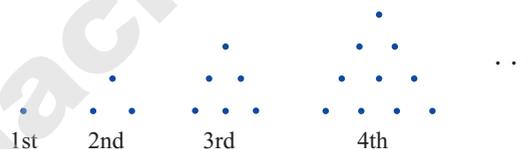
7. What digit is in the 99th decimal place in the decimal for $\frac{1}{7}$? Explain your reasoning.

8. a. What digit is in the ones place in the calculated form of 3^{250} ?
 b. What digit is in the ones place in the calculated form of 7^{350} ?
 c. What digit is in the ones place in the calculated form of $3^{250} \cdot 7^{350}$?
 d. Write down how you would proceed to find the digit in the ones place in the calculated form of 7^n for any whole number n .

9. The numbers $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, etc. are referred to as the square numbers. Tell how the drawings below explain why they are the *square* numbers.



10. The *triangular* numbers 1, 3, 6, 10, etc. are suggested by the numbers of dots in the following triangles of dots (the first one is not really a triangle):



- a. Make a table, such as the one that follows, for the first 8 triangular numbers, and look for a pattern.

Which triangle number?	1 st	2 nd	etc.
Number of dots	1	3	etc.

- b. Do the *drawings* support an argument that the pattern will continue?

11. A famous mathematician named Gauss (pronounced “gows”) was assigned the problem of adding the whole numbers from 1 through 100. Here is his clever approach:

$$\begin{array}{r} 1 + 2 + 3 + \dots + 98 + 99 + 100 \\ 100 + 99 + 98 + \dots + 3 + 2 + 1 \\ \hline 101 + 101 + 101 + \dots + 101 + 101 + 101 \end{array}$$

Notes

Learning Exercise 5: You may want to use these problems as discussion items in class. If so, here is some help: nature: lengths in nautilus shells, pine cones, and many other things; art: golden ratio [see (5c)]; geometry: ratios in a pentagon. Try typing “Fibonacci sequence” in a search engine such as Google.

Answers

a. 13, 21, 34, 55

b. The algebra gets more laborious for $n > 2$.

c. The ratios tend toward the value of the golden ratio, about 1.61803 (exactly $\frac{1+\sqrt{5}}{2}$).

Learning Exercise 10(a): If students do not notice the following pattern,

1

$$3 = 1 + 2$$

$$6 = 1 + 2 + 3$$

etc., you may want to help them out.

Notes

Then he reasoned: There are 100 of the 101s, or 10,100. But I added them twice, so $1 + 2 + 3 + \cdots + 98 + 99 + 100$ is half of 10,100, or 5050.

a. Apply this reasoning to the following sum:

$$1 + 2 + 3 + \cdots + (n - 2) + (n - 1) + n$$

b. How is part (a) relevant to Learning Exercise 10?

12. *Jump-It* is a label for this puzzle. There are 2 colors of round markers, 3 of each color. They are arranged on a 7-square row, as shown here.



a. Can you switch the colors so that each color is on the opposite end with an empty square in the middle, subject to these two rules?

Rule 1. A marker can be moved to an adjacent empty square.

Rule 2. A marker can “jump” a marker in the next square, if there is an empty square on the other side of the jumped marker.

- b. What is the minimum number of moves for the markers to change ends? (*Hint:* It is not 17.)
- c. What is the minimum number of moves if there are n markers on each end, with an empty square in the middle? (*Hints:* A good problem-solving approach is to look at simpler problems and look for a pattern. Here, there is a pattern relating the number on each end and the minimum number of jumps.)
13. On the first day of the month, a rich lady offers you two choices, and you must pick one:
- Choice I.* You can receive \$1000 at the end of the month.
- Choice II.* You get 1¢ if you stop after the first day, 2¢ if you stop after the second day, 4¢ if you stop after the third day, 8¢ if you stop after the fourth day, etc., until the end of the month.
- To get more money, which choice should you make? (*Hints:* $2^0 = 1$; $2^{10} = 1024$.)
14. An investment increases at a rate of 1.03 each year. If the original investment is \$500, what is the investment worth after 10 years?
15. When a pendulum swings freely, the length of its arc decreases geometrically. If the 4th arc is 16 ft and the 6th arc is 1 ft, what will the 5th arc be?

Learning Exercise 13: You may recognize this problem as a variant of a classical problem that often focuses on the sum.

Supplementary Learning Exercises for Section 12.2

1. Give for each list (1) the likely 500th number and (2) an expression for the likely n th number. If the sequence is arithmetic, identify a and d . If the sequence is geometric, identify a and r .
- | | |
|--------------------------------------|--|
| a. 8, 12, 16, 20, 24, ... | b. 2.3, 3.6, 4.9, 6.2, 7.5, ... |
| c. $-1, 1, -1, 1, -1, \dots$ | d. $3\frac{1}{4}, 5, 6\frac{3}{4}, 8\frac{1}{2}, 10\frac{1}{4}, \dots$ |
| e. $-100, -98, -96, -94, -92, \dots$ | f. 4, 7, 9, 4, 7, 9, 4, 7, 9, ... |
| g. 1, 5, 25, 125, 625, 3125, ... | h. 2000, 1998.3, 1996.6, 1994.9, 1993.2, ... |
| i. 3, 9, 27, 81, ... | j. $\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ |
| k. 16, 12, 8, 4, 0, ... | l. 0.5, 0.75, 1, 1.25, 1.5, ... |
2. a. Give the first five numbers in the arithmetic sequence that has a first number 22.3 and a difference of 0.35.
- b. Give the first five numbers of a geometric sequence that has a first number 1.5 and a ratio of 2.
3. What would the 36th digit be in the repeating pattern 3, 6, 9, 3, 6, 9, 3, 6, 9, ...? What would the 100th digit be?

4. What digit is in the 88th place (after the decimal point) of the decimal equivalent of $\frac{2}{11}$?
5. What digit is in the ones place in the calculated form of 2^{120} ?
6. Use the shortcut method from Activity 8 to find 85^2 and 19.5^2 .
7.
 - a. What digit is in the 38th place in the calculated form of $\frac{1}{11}$?
 - b. What digit is in the 29th place in the calculated form of $\frac{3}{7}$?
 - c. What digit is in the 43rd place in the calculated form of $\frac{1}{9}$?
8.
 - a. What is the n th term in the following sequence?

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$$

- b. What is the sum of the first n numbers in this sequence? That is, to what number is the sum getting closer and closer?

12.3 Functions and Algebra

The work with patterns can lead naturally to the idea of *function*, a topic that permeates mathematics, even if not made explicit. Informal work with functions in some guise can appear even in grades K–3.

For the sequence of numbers 3, 6, 9, 12, and 15, we note that each actual number is increased by 3 to get the next number. Each successive number in the sequence is generated from the preceding one. In contrast, you can think of the numbers in column 2 in the tables below as generated by multiplying the location in the list by 3, as the arrows in the tables suggest.

Location in the list	Actual number	n	s
1	→ 3	1	→ 3 = 3
2	→ 6	2	→ (3) + 3 = 6
3	→ 9	3	→ (3 + 3) + 3 = 9
4	→ 12	4	→ (3 + 3 + 3) + 3 = 12
5	→ 15	5	→ (3 + 3 + 3 + 3) + 3 = 15

The second table, with the column headings the variables n and s , should help you see that working with patterns can lead to an algebraic equation, such as $s = 3n$ for the pattern. Such an equation allows us to predict the number located in the 100th place in the sequence, just by multiplying by 3.

Tables like these, in which each value of n corresponds to exactly one value of s , incorporate the basic idea of a mathematical *function*. The dictionary usually gives the more common meanings for the word *function* before giving the mathematical meaning. Here is one definition as the word is used in mathematics.

A function from one set to another (which may be the same set) is a correspondence in which each element of the first set is assigned to exactly one element of the second set.

We can call the first set *inputs* and the second set *outputs*. In the most recent example, the input set (for n) consists of the whole numbers 1 through 5, and the output set (for s) would be any set containing 3, 6, 9, 12, and 15. In most cases in mathematics, there is a **function rule** that describes how the correspondence works. The function rule could be represented by the equation $s = 3n$, or by the words “multiply a number from the input set by 3 to get the number from the output set.”

Notes

A discussion of coordinate graphs begins in Chapter 13.

We switch here from recursive reasoning (a vertical search in a table) to looking for a function (a horizontal search in a table).

We do not include the terms *domain* and *range* because they are not used in elementary texts and are not needed here.

Notes

EXAMPLE 9

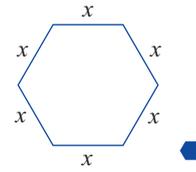
Write an equation for the perimeter of a regular hexagon with sides of length x inches. Use the variable p for the perimeter.

Solution

or even the longer equation

$$p = 6x,$$

$$p = x + x + x + x + x + x$$

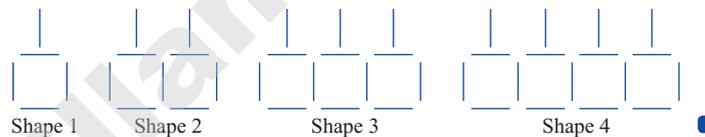


You may be familiar with notations such as $f(x)$, which denotes the value associated with x for some function f . The value associated with 2, for example, might be designated $f(2)$. The equation $f(x) = 6x$ could have been used in the last example, rather than $p = 6x$, to emphasize the functional relationship. Letters other than f can be used to signal a function, as in $g(x) = 2x + 3$ for some other situation. The context usually makes clear that $f(x)$ is not describing a product, as well as what numbers are acceptable for x and for $f(x)$. Although variables x and y are commonly used with functions, you may see equations such as $s = 16t^2$ or $f(t) = 16t^2$ in function contexts. Hence, from a more advanced mathematical viewpoint, you were also working with functions when you worked with patterns in the last section.

Let's revisit the toothpick pattern we looked at in Section 12.1, Activity 4. But this time we'll shift from thinking about how to build a shape based on the preceding one to associating shape number with the number of toothpicks. As you work through these types of problems, it may be helpful to record in your table how each shape has changed from the one before, as well as the total number of toothpicks needed to build the shape.

**ACTIVITY 9 Leah's Pattern Grows**

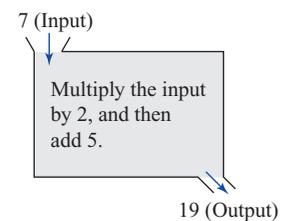
Look at the following sequence of shapes. In Activity 4, you described how you thought about building each shape given the shape before it. How many toothpicks does it take to build Shape 100? How many to build Shape n ?



The function rule you came up with describes how the shape number is related to the number of toothpicks needed to build it.

We can also represent functions using methods other than tables and function rules. *Dot diagrams* are illustrated in Activity 10. Sometimes, especially when no function rule is obvious, an ordered pair such as $(18, 25)$ is used to communicate the correspondence $18 \rightarrow 25$ (read as "18 is assigned to 25"). The first entry in the ordered pair is from the input set, and the second entry is from the output set. A *collection of ordered pairs* might then represent a function.

Some elementary school programs include not only work with patterns but also exposure to functions in the guise of *function machines*, like the one shown here. The words "input" and "output" might be used instead of x and y (or other variables) in tables. For example, the function rule $y = 2x + 5$ [or $f(x) = 2x + 5$] might not appear but could be expressed as "output equals two times the input, plus 5." Children might be asked to complete a table for such a machine.

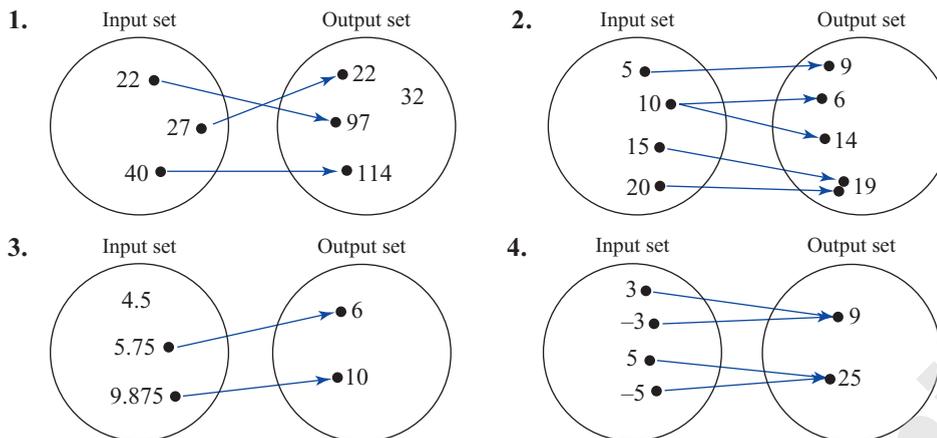


In the definition for *function*, it is important to note that *each* element in the first set corresponds to *exactly one* element in the second set. The table below does *not* describe a function, because 17 in the first set corresponds to two elements in the second set, 19 and 11.

Number from first (input) set	17	15	17	21
Number from second (output) set	19	23	11	13


ACTIVITY 10 Which Correspondences Give Functions?

Tell whether each of the following diagrams or problems describes a (mathematical) function, and explain how you know. Notice the various ways to show functions.



5. To each postage stamp, assign the weight of the first-class letter the stamp will pay for. For example, a 45¢ stamp will pay for weights up to and including 1 oz.
6. The collection of ordered pairs: (0, 1), (1, 2), (2, 3), (3, 4), etc.
7. The collection of ordered pairs: (2, 5), (3, 8), (2, 9), (5, 10)
8. Every number is paired with its square, so 1 is paired with 1, $\frac{1}{2}$ is paired with $\frac{1}{4}$, $1\frac{2}{3}$ is paired with $\frac{25}{9} = 2\frac{7}{9}$, $\sqrt{2}$ is paired with 2, etc.
9. In any classroom, associate with each first name a student's full name. Here is part of the class roster for a particular classroom.

First Name	Full Name
Emily	Emily Dickinson
Memo	Memo Flores
Rachel	Rachel Hobbs
Emily	Emily Bronte
Jon	Jon Sinclair

The restrictive phrase *exactly one* in the definition of function assures that there is no ambiguity in determining what corresponds to a specific element of the input set. With a table of given values and their corresponding values, you can do a “What’s my rule?” activity, in which you look for a pattern and express its function rule.

THINK ABOUT ...

“What’s my rule” for the following data?

When you say ...	25	16	40	3
I say ...	75	66	90	53

Using a table to find a function rule is a good practice, but doing so has risks. One risk is that even though you might think you found the function rule, it might not be the correct one. For example, the table in the preceding Think About “obviously” suggests the equation $y = x + 50$, but the function rule might actually be something else.

Notes
ACTIVITY 10
Answers

1 and 4. Diagrams 1 and 4 display functions. Each number in the first set is matched with exactly one number in the second set.

2 and 3. Diagram 2 contains a number in the first set matched with more than one number in the second set. Diagram 3 contains an element in the first set that is not matched with any element in the second set.

5. This is not a function. For example, a 45¢ stamp would be matched with both 0.3 oz and 0.4 oz.

6. This is a function.

7. This is not a function because the number 2 is matched with both 5 and 9.

8. This is a function. Each number has only one square.

9. This is not a function. Emily would be matched with two different full names.

MathClips

▶▶ I What’s My Rule?

SEE INSTRUCTOR NOTE 12.3A on Occam’s razor, which is a “rule” for when to trust a generalization.

Notes

EXAMPLE 10

Consider this table for a “What’s my rule?” problem:

x	y
0	1
1	2
2	4

Do you see that both $y = 2^x$ and $y = \frac{1}{2}x^2 + \frac{1}{2}x + 1$ describe the table entries? ◆

If the table is about some real situation, then you can check other values for x . But with just a few numbers to work with, you cannot be certain that *your* rule is the only possible

Try these “What’s my rule?” exercises from a sixth-grade textbook.

Find the price of one ticket, p , when 3 tickets cost \$16.50.

$$3p = \$16.50$$

$$\frac{3p}{3} = \frac{\$16.50}{3}$$

$$p = \$5.50$$

One ticket costs \$5.50. Check the cost for 4 and 5 tickets.

$$4 \times \$5.50 = \$22.00$$

$$5 \times \$5.50 = \$27.50$$

\$5.50 checks for 4 and 5 tickets.

State the rule:
The total cost, c , is \$5.50 times the number of tickets, n .

Write an equation:
 $c = \$5.50 \times n$, or $c = 5.5n$

Find the cost of 6 tickets:
 $c = 5.5(6)$
 $c = 33$
The cost of 6 tickets is \$33.00.

Write a rule and an equation to fit a pattern in each table for **9** and **10**. Then use the rule to complete the table.

9.

u	0	1	2	3	4
v	25	24	23		

10.

x	0	9	18	27	36
y	0	1	2		

Problem Solving MATHEMATICAL PRACTICES

11. To celebrate their 125th anniversary, a company in Germany produced 125 very expensive teddy bears. The bears, known as the “125 Karat Teddy Bears,” are made of mohair, silk, and gold thread and have diamonds and sapphires for eyes.

The chart at the right shows the approximate cost of different numbers of these bears. Based on the pattern, how much does one bear cost?

Cost of “125 Karat Teddy Bears”	
Number, n	Cost, c
4	\$188,000
7	\$329,000
11	\$517,000
15	\$705,000

12. Writing to Explain Explain how you can find a pattern in the chart showing the cost of “125 Karat Teddy Bears.” Use the pattern to write a rule and an equation.

13.

r	2	4	6	8	10	12
s	0	2	4	6	8	10

A $s = r + 2$ **C** $s = 2r$
B $s = r - 2$ **D** $s = r$

14.

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$
y	0	1	2	3	4	5

A $y = x + 2$ **C** $y = 2x$
B $y = x$ **D** $y = 3x$

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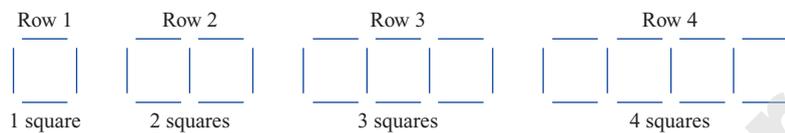
one. At some stage in their mathematics education, children should know that trusting a generalization based on a pattern is risky, but most work in schools proceeds as though a rule is *the* rule. Arriving at a general assertion by looking at examples is called **inductive reasoning**. Example 10 demonstrates that inductive reasoning might produce good ideas but is risky, as more than one interpretation may be appropriate.

Rather than trust completely a function rule or pattern based on a table of values, the mathematically sound way to proceed is to look for a justification that the rule must be correct. We will use the problem given in Activity 11 to illustrate what a possible justification might look like in Discussion 2, which follows shortly. Note that the use of a specific large number, such as 100 in Activity 11, means that a child does not have to be algebraically sophisticated to *reason* about the situation.



ACTIVITY 11 Making a Row of Squares from Toothpicks

Pat made a pattern of squares from toothpicks, as in this drawing:



How many toothpicks would Pat need to make a row with 100 squares? More generally, how many toothpicks would it take to make a row with n squares? ■

Your table in Activity 11 probably showed that the row with 1 square requires 4 toothpicks, the row with 2 squares requires 7 toothpicks, etc. Did you see a pattern that enabled you to predict that it would take 301 toothpicks for the row with 100 squares and $3n + 1$ for the row with n squares? The results for 100 and n certainly seem to fit the pattern. But is there a risk in endorsing these results enthusiastically? Some other rule might apply, perhaps as the number of squares gets much larger than 100. Is there any way to argue that the results will always be correct for all values of n ? The answer happens to be *yes*.

When dealing with patterns in mathematics, we try to justify the results in some way to show that the result *must be* true in *all* cases, not just the cases examined. Giving a justification confirms that a particular generalization from inductive reasoning is all right.



DISCUSSION 2 Are These Justifications Convincing?

Which of these arguments, used to justify the result for the row with 100 squares, could be modified to justify the general $3n + 1$ result for a row of n squares from Activity 11? If an argument is not strong, explain why.

1. *Mike*: “I needed 4 toothpicks for the first square, and then 3 more for each of the 99 squares after that. I got 301.” In general, Mike’s argument suggests the rule $f(n) = 4 + (n - 1)3$ for n squares.
2. *Nadia*: “I checked two more cases, and they worked. So it has to be all right.”
3. *Oscar*: “Well, I put down 1 toothpick, and then every time I put down 3 more, I got another square. So $1 + 100 \times 3$.” In general, Oscar’s reasoning suggests the rule $f(n) = 1 + 3n$.
4. *Paloma*: “Across the top, there would be 100 toothpicks. In the middle, 101. And on the bottom, 100. So, 301.” In general, Paloma’s reasoning suggests the rule $f(n) = n + (n + 1) + n$.
5. *Quan*: “100 squares would take 400 toothpicks by themselves. But when you put them together, you don’t need the extra ones inside, so subtract 99.” In general, Quan’s reasoning suggests the rule $f(n) = 4n - (n - 1)$.
6. *Ricky*: “Everyone I asked said the same thing, 301.” ■

Notes

ACTIVITY 11

If your students do not notice that the sequence 4, 7, 10, . . . is an arithmetic sequence, then point this out.

Answers

A row with 100 squares would require 301 toothpicks. A row with n squares would require $4 + 3(n - 1)$, or $3n + 1$ toothpicks.

DISCUSSION 2

You should have some good discussions with these answers. Students will be surprised by the various ways to justify that $3n + 1$ toothpicks are needed.

Answers

Nadia’s answer is weak because one cannot generalize from two cases. Ricky’s answer is weak because everyone he asked could be wrong.

Notes

Because the toothpick problem in Activity 11 is based in reality, there is a good chance that a mathematically sound justification can be offered, as was illustrated by several of the arguments given in Discussion 2. If the table were just about numbers (with no context given), there would be no way to know whether the rule that you find (or an algebraic equivalent) is true in all cases.

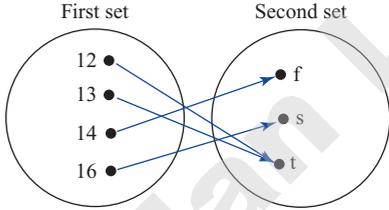
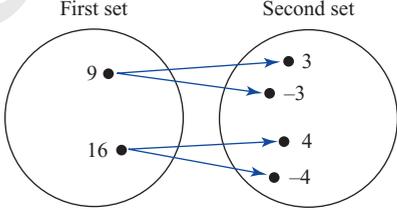
TAKE-AWAY MESSAGE . . . The study of patterns is the basis for the more advanced study of mathematical functions. Many numerical patterns can be described by general function rules that enable us to determine the actual number, given any location in the pattern. Functions may also be represented in tables or with dot diagrams. Although a pattern based only on examples (inductive reasoning) may have more than one interpretation, many times it is possible to find a justification showing that the pattern will be true given a context.

Learning Exercises for Section 12.3

Learning Exercises 12.3

Students do not have answers or hints for Learning Exercises 2(b, d), 3, 5, 7–10, 12(a–b, e), 13(b–c), 14, 15, and 17–21. Be sure to assign Learning Exercises 2 and 13. At least some parts of Learning Exercise 12 should be assigned to give practice at justifying.

- In a classroom of children, do the following assignments give functions? Explain why or why not.
 - To each child, assign his or her first name.
 - From a list of the last names of children in the class, assign the child.
- Which of the following dot diagrams or ordered pairs represent a function? How do you know?

a.  b. 

c. $(8, 56), (11, 77), (9, 63), (0, 0)$ d. $(1.7, 2), (1.8, 2), (3.2, 3), (4.3, 4)$

In Learning Exercises 3–10, find a possible function rule for each table. Then use your rule to find the numerical value for n , if n is indicated. (*Hint: Do any of them give arithmetic sequences?*)

3.

x	1	2	3	4	...	n
y	9	30	51	72	...	1038

4.

x	1	2	3	4	...	n
$f(x)$	78	56	34	12	...	-120

5.

input	1	2	3	4	...	n
output	0	8	16	24	...	1608

6.

input	50	51	52	53	...	n
output	8	8.3	8.6	8.9	...	15.8

7.

x	3	1	4	2	...	n
y	12	2	17	7	...	547

Learning Exercises 7, 8, and 10: It is likely that these problems will require new tables, with the first-row entries ordered or augmented.

8.

x	2	1	3	0	4	...	n
$f(x)$	35	19	51	3	67	...	403

9.

x	0	1	2	3	4	...	n
$f(x)$	$\frac{1}{2}$	$1\frac{1}{6}$	$1\frac{5}{6}$	$2\frac{1}{2}$	$3\frac{1}{6}$...	$8\frac{1}{2}$

10.

x	0	3	4	5	...	n
y	4	8	$9\frac{1}{3}$	$10\frac{2}{3}$...	84

11. Find a function rule for each table.

a.

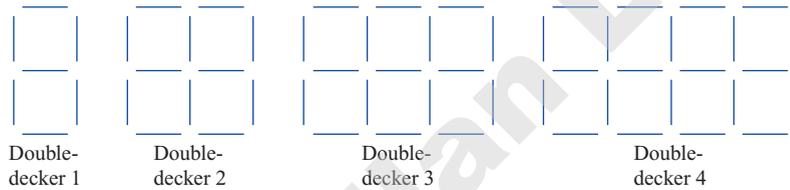
x	1	2	3	4	...	n
$g(x)$	1	4	9	16	...	900

b.

x	1	2	3	4	...	n
$g(x)$	1	8	27	64	...	729

12. Find a function rule for each of the following patterns, and justify that your rule will be true in general.

a. the number of toothpicks to make Double-decker n :



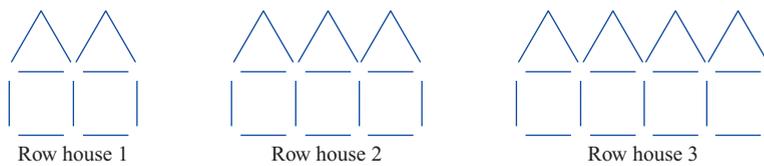
b. the number of toothpicks to make Shape n :



c. the number of toothpicks to make Shape n :



d. the number of toothpicks to make Row house n :



e. Make up a toothpick pattern of your own, and challenge others to find a rule for it. Can you justify the rule?

Notes

Learning Exercise 11: Note that this problem involves squares and cubes.

Notes

13. Some *sums* of patterns can be predicted.

a.

How many evens (starting with 2)?	Sum of the evens
1	2
2	$2 + 4 = \mathbf{6}$
3	$2 + 4 + 6 = \mathbf{12}$
4	$2 + 4 + 6 + 8 = \mathbf{20}$
5	$2 + 4 + 6 + 8 + 10 = \mathbf{30}$
n	$2 + 4 + \dots + (2n) = ?$

b. Use part (a) and algebra to show that

$$1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$$

c. Use part (b) and algebra to show that

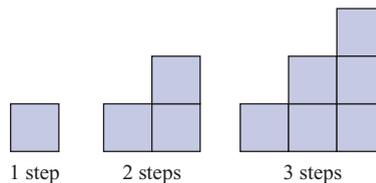
$$6 + 12 + 18 + 24 + \dots + (6n) = 3n(n+1)$$

d. What is the sum of the first n odd numbers? [Hint: Make a table, or use your results from parts (a) and (b).]

14. Rather than just finding the n th number in an arithmetic sequence, many times we want to find the *sum* of the first n numbers. Look for a pattern in the following table to see whether the sum of the first n terms in an arithmetic sequence is predictable. [Hint: See Learning Exercise 13(b).]

How many?	Arithmetic sequence	Sum
1	a	a
2	$a + d$	$a + (a + d) = 2a + d$
3	$a + 2d$	$a + (a + d) + (a + 2d) = 3a + 3d$
4	$a + 3d$	$\dots = 4a + 6d$
5	$a + 4d$	$\dots = 5a + 10d$
n	$a + (n-1)d$	$\dots = ?$

15. How many small squares will be in an n -step stairway like those shown in the following diagram? [Hint: See Learning Exercise 13(b).]



16. Formulas can be viewed as defining functions. For example, the formula $C = \frac{5}{9}(F - 32)$ tells what Celsius temperature, C , corresponds to a Fahrenheit temperature, F .

- a. What Celsius temperatures correspond to the boiling point of water (212°F at sea level), the freezing point of water (32°F), normal body temperature (98.6°F), and a comfortable room temperature of 70°F ?
- b. Use algebra to write the Fahrenheit temperature F as a function of the Celsius temperature C : $F = \underline{\hspace{2cm}}$.

17. What digit is in the 1000th decimal place in the decimal for $\frac{13}{303} = 0.042904290429 \dots$?

18. a. What digit is in the ones place in the calculated form of 8^{999} ?
- b. What digit is in the ones place in the calculated form of 16^{423} ?
- c. What digit is in the ones place in the calculated form of $8^{999} + 16^{423}$?

19. Each of the three students below is arguing that his or her function rule is the correct one. What do you say to them?

Abe: $f(x) = 2(x + x)$

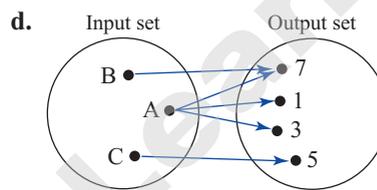
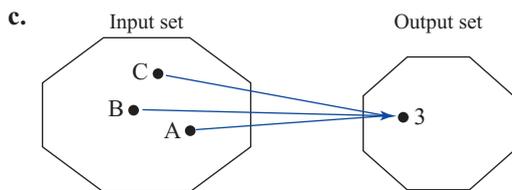
Beth: $f(x) = 4x$

Candra: $f(x) = x + x + x + x$

20. Many function rules that appear in elementary school will be simple, but polynomial rules like $f(n) = 2n^3 + 3n + 8$ can occur. The variable can also appear in exponents, as in geometric sequences with ar^{n-1} . Verify for $n = 1, 2, 3, 4, 5$, and 6 that $f(n) = (-1)^n$ gives the n th number in the list $-1, 1, -1, 1, -1, \dots$

Supplementary Learning Exercises for Section 12.3

1. Which of the following rules, diagrams, or ordered pairs are examples of functions?
- a. Passengers on a plane in the first set, matched with seats on the airplane as the second set; each passenger gets only one seat.
 - b. Seats on an airplane in the first set, matched with passengers on the airplane in the second set.



- e. Every square number is matched with its square roots.
 - f. Every whole number is matched with its square.
 - g. $(8, 3), (7, 4), (6, 3), (5, 2), (4, 1), (3, 0), (2, 1), (0, 2)$
 - h. $(3, 8), (4, 7), (3, 6), (2, 5), (1, 4), (0, 3), (1, 2), (2, 0)$
2. Find a likely function rule for each of the following tables, and find the missing values for m and n :

a.

x	y
1	12
2	17
3	22
4	27
\vdots	\vdots
100	m
n	2007

b.

x	$f(x)$
1	153
2	149
3	145
4	141
\vdots	\vdots
36	m
n	-43

c.

input	output
1	5
2	7
3	11
4	19
\vdots	\vdots
10	M
n	2051

d.

input	output
10	35
11	38
12	41
13	44
\vdots	\vdots
100	m
n	641

e.

x	$g(x)$
1	2
2	9
3	28
4	65
\vdots	\vdots
100	m
n	8001

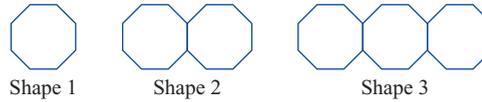
f.

x	$h(x)$
3	18
1	4
4	25
2	11
\vdots	\vdots
25	m
n	2454

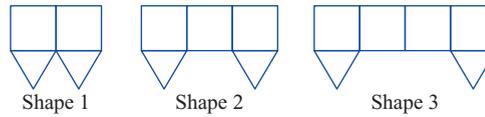
Notes

3. Find a function rule for each of the following shapes, and justify that your rule will be true in general:

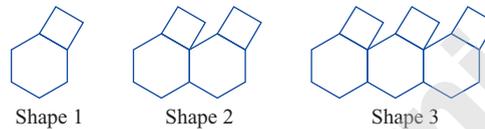
- a. the number of toothpicks to make Shape n in the pattern:



- b. the number of toothpicks to make Shape n in the pattern:



- c. the number of toothpicks to make Shape n in the pattern:



4. How are an arithmetic sequence and a geometric sequence alike? How are they different?
5. Continue each of the following sequences for six more terms, using the given information:
- arithmetic; first number 12, common difference 27
 - arithmetic; first number 3, common difference $6\frac{3}{4}$
 - geometric; first number 3, common ratio 5
 - geometric; first number 8, common ratio $\frac{2}{3}$
6. What would be the number of cricket chirps per minute when the temperature outside is 90°F if the pattern in the following table stays the same? Find a likely function rule.

Number of chirps	144	152	160	168	176
Temperature ($^\circ\text{F}$)	76	78	80	82	84

12.4 Algebra as Generalized Arithmetic

This section treats one prominent view of algebra: algebra as generalized arithmetic. In other words, what we've done so far in arithmetic with numbers we can now do in algebra with variables. For example, consider the properties of operations reviewed in Section 12.1 using symbols. Symbols are first learned as properties of arithmetic operations and are then generalized and used in algebraic operations. Let's explore this further.

EXAMPLE 11

We can think of $7x + (3x + 19)$ as $(7x + 3x) + 19$ using associativity of addition, and then we can use the distributive property to write $(7 + 3)x + 19$ and arrive at $10x + 19$. Indeed, if only addition is involved, grouping symbols are often omitted, except perhaps to add clarity or emphasis. Thus, $9x + (39 + x^2)$ can be replaced by $x^2 + 9x + 39$. The sequence of steps using addition properties that make this replacement possible is as follows:

$$\begin{aligned}
 9x + (39 + x^2) &= 9x + (x^2 + 39) && \text{using the commutative property} \\
 &= (9x + x^2) + 39 && \text{using the associative property} \\
 &= (x^2 + 9x) + 39 && \text{using the commutative property} \\
 &= x^2 + 9x + 39 && \text{because the associative property tells us we} \\
 &&& \text{do not need parentheses when adding} \\
 &&& \text{three numbers} \bullet
 \end{aligned}$$

This section features some parallels between arithmetic and algebra and provides further review of properties of operations.

Some states require an exposure to polynomial arithmetic on the part of elementary school teachers, so this section offers some work along those lines.

SEE INSTRUCTOR NOTE 12.4A, which points out the importance of making the connection between arithmetic and algebra.

EXAMPLE 12

We can say that $y(4y^2) = (4y)y^2 = 4(y \cdot y^2) = 4y^3$ because of the commutative and associative properties of multiplication. We usually would say simply that $y(4y^2) = 4y^3$, but it is important to recognize which properties allow us to arrive at this equality. ●

By combining these properties with what we know about order of operations, we can perform many algebraic manipulations and solve algebraic equations. (As a reminder, first perform operations within grouping symbols; then attend to exponents; then perform all multiplications and divisions moving from left to right; and finally, perform all additions and subtractions moving from left to right.)

EXAMPLE 13

Simplify the following expressions:

- a. $3 + 11(15 - 3) - 12 \div 4 = 3 + 11 \cdot 12 - 3 = 11 \cdot 12 = 132$
 b. $2x(5x - 13) + 3x^2 - x(12x - x - 13)$
 $= 10x^2 - 26x + 3x^2 - x(11x - 13)$
 $= 13x^2 - 26x - 11x^2 + 13x = 2x^2 - 13x$ [Recall: $-a(-b) = ab$.] ●

Our place-value numeration system gives another illustration of algebra as a generalization of arithmetic. You know that one expanded form of 452 is $400 + 50 + 2$, or $4 \cdot 100 + 5 \cdot 10 + 2$. A similar expanded form uses exponents: $452 = 4 \cdot 10^2 + 5 \cdot 10^1 + 2 \cdot 10^0$. Because $a^0 = 1$ for any nonzero value of a , we know that $10^0 = 1$, so we can use $4 \cdot 10^2 + 5 \cdot 10 + 2$ for the expanded form.

If we use the variable x instead of 10 in the last expression, we get the expression $4 \cdot x^2 + 5 \cdot x^1 + 2$, or $4x^2 + 5x + 2$, which is a *polynomial in x* .

A **polynomial in some variable** is any sum of number multiples of whole number powers of the variable. The expressions that are added (or subtracted) are called **terms** of the polynomial, so $4x^2$, $5x$, and 2 are the terms of the polynomial $4x^2 + 5x + 2$.

Arithmetic that has been learned conceptually can lead naturally to polynomial arithmetic. Addition of polynomials is very similar to addition of multidigit whole numbers.

EXAMPLE 14

	Consider	and contrast that with
	$\begin{array}{r} 452 \\ +324 \\ \hline \end{array}$	$\begin{array}{r} 4x^2 + 5x + 2 \\ + 3x^2 + 2x + 4 \\ \hline \end{array}$
The parallel is clearer in the expanded form:	$\begin{array}{r} 4 \cdot 10^2 + 5 \cdot 10 + 2 \\ + 3 \cdot 10^2 + 2 \cdot 10 + 4 \\ \hline \end{array}$	$\begin{array}{r} 4x^2 + 5x + 2 \\ + 3x^2 + 2x + 4 \\ \hline \end{array}$
The usual algorithm for adding whole number transfers to the polynomial form:	$\begin{array}{r} 4 \cdot 10^2 + 5 \cdot 10 + 2 \\ + 3 \cdot 10^2 + 2 \cdot 10 + 4 \\ \hline 7 \cdot 10^2 + 7 \cdot 10 + 6 \end{array}$	$\begin{array}{r} 4x^2 + 5x + 2 \\ + 3x^2 + 2x + 4 \\ \hline 7x^2 + 7x + 6 \end{array}$ ●

The usual algorithms are efficient in part because they ignore the many steps that are necessary if one had to be explicit about the properties of operations involved. The uses of the properties are clearer when the calculation is written in horizontal form.

Notes

See Appendix F: A Review of Some Rules for more on order of operations, using exponents, and using signed numbers. In particular, note the last simplification in part (b) of Example 13.

Notes

A review of the properties may be in order if you skipped Section 12.1.

ACTIVITY 12

Answers

- a. $(16 - \frac{3}{4})(12 - 8) + 9$
 $= 15\frac{1}{4} \times 4 + 9$
 $= 61 + 9$
 $= 70$, or $16 \times 4 - \frac{3}{4} \times 4 + 9$
 $= 64 - 3 + 9$
 $= 70$
- b. Direct substituting leads to -735 .
- c. $3t - 3rt - 27r - 6r^2 = -735$

THINK ABOUT . . .

Many examples with different values could be given. Here is one: If $s = r$, then $s^2 = r^2$. Note that the converse is not necessarily true.

**DISCUSSION 3** Properties Working for Us

$452 + 324$ is the same as $(4 \cdot 10^2 + 5 \cdot 10 + 2) + (3 \cdot 10^2 + 2 \cdot 10 + 4)$. When placed in vertical form as shown at right, the algorithm usually is computed from right to left: $(2 + 4) + (5 + 2) \cdot 10 + (4 + 3) \cdot 10^2$, in that order. (In fact, the place values are often ignored, and the calculations done are $2 + 4$, $5 + 2$, and $4 + 3$. When this happens, children lose awareness of the place value implicit in the numbers.) In the horizontal form, however, it is more natural to work from left to right: $(4 + 3) \cdot 10^2 + (5 + 2) \cdot 10 + (2 + 4)$. What properties assure that

$$(4 \cdot 10^2 + 5 \cdot 10 + 2) + (3 \cdot 10^2 + 2 \cdot 10 + 4)$$

is indeed the same expression as

$$(4 + 3) \cdot 10^2 + (5 + 2) \cdot 10 + (2 + 4) \blacksquare$$

Isn't it fortunate that the usual algorithm for adding multidigit whole numbers allows us to bypass being explicit about each use of a property?

**ACTIVITY 12** Time to Practice

- a. Evaluate $16(12 - 8) - \frac{3}{4}(12 - 8) + 9$ in two different ways, using the distributive property and then using a second method.
- b. Evaluate $3t + r(t - 7) - 4r(t + r + 5) - 2r^2$ when $r = 7$ and $t = 14$.
- c. Simplify $3t + r(t - 7) - 4r(t + r + 5) - 2r^2$ first, and then evaluate your new expression when $r = 7$ and $t = 14$. How does your answer compare with that for part (b)? \blacksquare

**Focus on SMP MP7**

Mathematically proficient students look for and make use of structure. When a student interprets mathematical expressions, it is not easy initially to see the structure. Seeing the structure in an expression can be challenging. Consider $3t + r(t - 7)$. What support can we give students for seeing the structure?

$$\begin{array}{c} \text{-----Sum-----} \\ 3t + r \cdot (t - 7) \\ \text{Difference} \\ \underbrace{\hspace{2cm}} \\ \text{Product} \end{array}$$

Symbols are also used to express many other relationships that are true for any numbers substituted properly. Here are examples of more properties that are frequently used in algebra but that do not have formal names:

- For any number a , $a \cdot 0 = 0 \cdot a = 0$.
- If $a > b$ and $c > d$, then $a + c > b + d$ for any values of a , b , c , and d .
- $(pq)^2 = p^2q^2$ for any values p and q .
- $a + c - c = a$ for any values a and c .
- $(x + y)^2 = x^2 + 2xy + y^2$ for any values x and y .

THINK ABOUT . . .

Are you convinced that the properties described in the preceding list hold for all possible values of a , b , c , d , p , q , x , and y ? Choose some values and test these claims. Can you write down another property using s and t that is true for all values of s and t ?

Algebra problems are often expressed as *equations* for which we seek values for variables that will lead to true statements. A simple example would be $3 + n = 28$. When 25 is substituted for n , the statement is true. We say that we have *solved the equation* $3 + n = 28$, and we call 25 the *solution* to this equation. If any other value is substituted for n in the equation $3 + n = 28$, the statement is false. Some equations may have more than one value that will give a true statement. For example, $x^2 = 4$ has two solutions, 2 and -2 , because when we substitute either value for x , we get a true statement. We say that 2 and -2 are solutions to the equation $x^2 = 4$.

EXAMPLE 15

Which of the values $-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$ can be substituted for the variable in each equation to make true statements?

- a. $j + 8 = 13$ Only the value 5 makes this statement true.
 b. $k + 8 = 8$ Only the value 0 makes this statement true.
 c. $n^2 = 9$ Only the values 3 and -3 make this statement true. ●

Obviously, substituting values for variables in an equation, which is a trial-and-error method, is not an efficient way of solving an equation. Instead, we rely on four more properties of equalities to solve equations.

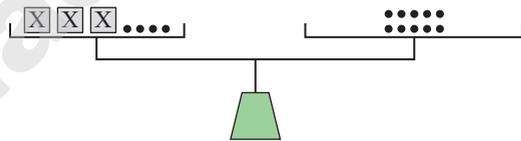
Following are four important properties of equalities for any given values a, b , and c :

1. If $a = b$, then $a + c = b + c$.
2. If $a = b$, then $a - c = b - c$.
3. If $a = b$, then $a \cdot c = b \cdot c$.
4. If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.

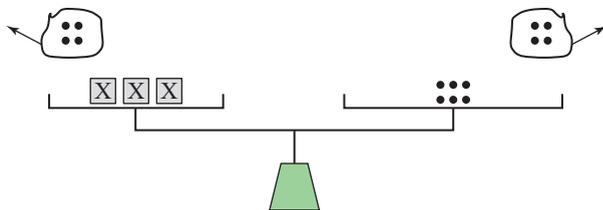
Basically, these properties tell us that whatever operation we perform on the left side of the equal sign must also be performed on the right side of the equal sign (and vice versa), to assure that equality is maintained.

A balance model (pictorial or physical) is often used to reinforce the idea of equality. Recall from our discussion of children's early algebra that young children can see the equal sign as a signal to operate, rather than as a sign of equality of two sides. When children use a balance model, they can model the properties of equality when solving equations.

Here is a balance display that models the equation $3x + 4 = 10$.



The balance can be used to model the subtraction of 4 from each side (i.e., by showing the removal of 4 dots from each side), illustrating that $3x + 4 - 4 = 10 - 4$, or $3x = 6$. Furthermore, if $3x$'s balance with 6 units, then each of the x 's must be equal to 2 units.



Notes

Another major topic of this section is solving equations by treating both sides of an equality the same way. This discussion of solving linear equations is brief—we expect that most students will remember this process. Solving quadratic equations goes beyond the basic ideas here. You will want to assess how much students may have forgotten.

You may need to discuss why division by 0 is not possible. This topic was first discussed in Section 3.4.

Notes

EXAMPLE 16

Solve each of the following equations. Describe which properties you used. The balance imagery may be useful.

- a. $10n - 14 = 22 - 8n$ b. $1.4x + 5 = 16.2$ c. $3g - 7 + 2g = 13$
 d. $\frac{2}{3}x = 18$ e. $2t + 125 = 9t + 20$

Solution

- a. The steps for part (a) are provided in more detail than are the steps for parts (b–e).
 (i) Using the first property of equality listed, we can add 14 to both sides of the equation: $10n - 14 + 14 = 22 - 8n + 14$.
 (ii) On the left side, subtracting 14 from $10n$ is like adding -14 . Because $-14 + 14 = 0$ by the additive inverse property, the left side becomes just $10n$. On the right side, we can combine 22 and 14. The result is $10n = 36 - 8n$.
 (iii) Next, we again use the first property of equality and add $8n$ to both sides: $10n + 8n = 36 - 8n + 8n$.
 (iv) Using the distributive property on the left and the additive identity property on the right leaves us with $18n = 36$.
 (v) Finally, using the fourth property of equality, we can divide both sides by 18 to arrive at $n = 2$.
 b. Subtract 5 from *both* sides to get $1.4x = 11.2$. Then divide both sides by 1.4: $\frac{1.4x}{1.4} = \frac{11.2}{1.4}$. So $x = 8$ is the solution if there are no calculation mistakes. You can check by substituting 8 for x in the original equation to see whether the equation statement is true. Is $1.4(8) + 5 = 16.2$?
 c. Working just on the left side, we can change the original equation to $5g - 7 = 13$. Then we add 7 to each side, to get $5g = 20$. Dividing both sides by 5 gives $g = 4$.
 d. Divide both sides by $\frac{2}{3}$ (or multiply both sides by $\frac{3}{2}$), to get $x = 27$.
 e. You can arrive at the answer in the form of “some number = t .” With $9t$ on the right side and $2t$ on the left side, we will isolate the t on the right side. Subtract $2t$ and 20 from each side, giving $105 = 7t$. Dividing both sides by 7 gives $15 = t$ as the solution. ●

THINK ABOUT ...

For $10n - 14 = 22 - 8n$, substitute 2 in place of n in every step along the way. What happens? For $5x - 3 = 2x$, is it also true (but not so useful for solving the original equation) that $5x - 3 + 500 = 2x + 500$?

Some students find it helpful to think of the process of solving an equation as “uncovering” to find the value of the unknown. We uncover when we use properties to “isolate” the variable on one side of the equation.

**ACTIVITY 13 Seeking the Truth**

- Solve these equations, and tell what properties you used:
 - $5z + 14 = 7z - 30$
 - $19 + 13t = 28t - 56$
- Write an equation with a variable for each of the following elementary school problems, and then solve the equation:
 - Ana has to practice the piano for 15 minutes every day. Today she has practiced 7 minutes. How many more minutes does she need to practice today?

THINK ABOUT ...

Not all practicing teachers realize that the value for the unknown quantity makes true every statement in each stage of the process of solving an equation, nor do they understand that properties are used only when they lead toward a solution.

ACTIVITY 13

Answers

- $z = 22$
 - $t = 5$
- $n + 7 = 15$; $n = 8$
 - \$6.57
 - \$1.60

Note: Students often divide when solving problems like 2(c)—they think they should divide because the answer must be smaller than \$2.19. This error was discussed in Chapter 3 of Part I.

- 8400 mm^2
- 11 ft

- b. One kind of cheese costs \$2.19 a pound. How much will a package weighing 3 pounds cost?
- c. One kind of cheese costs \$2.19 a pound. How much will a package weighing 0.73 pound cost?
- d. A rectangle has a height of 70 mm and a base of 120 mm. What is the area of the rectangle?
- e. A triangle with an area of 44 ft^2 has a base of 8 ft. What is its height? (Recall that the formula for the area of a triangle is $A = \frac{1}{2}bh$.) ■

The usual algorithm for multiplying multidigit whole numbers also disguises the fact that properties can explain why the algorithm gives correct answers. The properties also apply to the multiplication of polynomials.

DISCUSSION 4 Multiplying Polynomials Is Like Multiplying Whole Numbers

1. How does $\begin{array}{r} 32 \\ \times 4 \\ \hline \end{array}$ transfer to $\begin{array}{r} 3n + 2 \\ \times 4 \\ \hline \end{array}$?

What property or properties are involved?

2. How does $\begin{array}{r} 32 \\ \times 14 \\ \hline \end{array}$ transfer to $\begin{array}{r} 3n + 2 \\ \times n + 4 \\ \hline \end{array}$?

What property or properties are involved? ■

Writing the calculations in Discussion 4 in horizontal form helps you see what properties are being used. For the numerical calculation:

$$4(30 + 2) = (4 \times 30) + (4 \times 2) \quad (\text{distributivity})$$

And for the corresponding algebraic calculation:

$$4(3n + 2) = (4 \times 3n) + (4 \times 2) \quad (\text{distributivity})$$

Thus, distributivity (of multiplication over addition) is involved in the numerical calculation as well as in the algebraic calculation. Furthermore, $(10 + 4) \times 32 = (10 \times 32) + (4 \times 32)$ does not give all the steps in the usual algorithm until we use distributivity again:

$$(10 \times 32) + (4 \times 32) = (10[30 + 2]) + (4[30 + 2])$$

This equation can also be expressed as follows, but not in the usual algorithm order because that algorithm starts at the right:

$$(10[30 + 2]) + (4[30 + 2]) = (10 \times 30) + (10 \times 2) + (4 \times 30) + (4 \times 2)$$

We compute the four multiplications shown on the right side of the equation when multiplying 14 and 32.

Similarly, $(n + 4)(3n + 2)$ involves the sum of the four multiplications $4 \cdot 2$, $4 \cdot 3n$, $n \cdot 2$, and $n \cdot 3n$. (Compare the vertical form.) Notice that each term in the $n + 4$ expression is multiplied by each term in the $3n + 2$ expression. In either the numerical or the algebraic case, distributivity is used more than once.

DISCUSSION 5 Algebraic and Numerical Division Algorithms

How does $12 \overline{)276}$ transfer to $x + 2 \overline{)2x^2 + 7x + 6}$? ■

Thus, operations on polynomials are similar to operations on numbers. Operations on algebraic fractions are also similar to operations on arithmetic fractions.

Notes

DISCUSSION 4

Answers

1. The distributive property is used: $4(3n + 2) = 12n + 8$.

2. The distributive property is used twice:

$$\begin{aligned} (n + 4)(3n + 2) &= n(3n + 2) + 4(3n + 2) \\ &= 3n^2 + 2n + 12n + 8 \\ &= 3n^2 + 14n + 8 \end{aligned}$$

This explanation follows the discussion.

Multiplication of binomials is mentioned, but for the most part we do not discuss factoring; that would require a great deal of review on the part of most students.

DISCUSSION 5

Think of 12 as $10 + 2$ and 276 as $2 \cdot 10^2 + 7 \cdot 10 + 6$.

Notes

EXAMPLE 17**With numbers**

Suppose we want to add the numeric fractions $\frac{5}{16} + \frac{7}{72}$.

We first need to find a common denominator, preferably the least common denominator. Because 16 is 2^4 and 72 is $2^3 \cdot 3^2$, the LCD is $2^4 \cdot 3^2$.

$$\begin{aligned}\frac{5}{2^4} + \frac{7}{2^3 \cdot 3^2} &= \frac{5 \cdot 3^2}{2^4 3^2} + \frac{7 \cdot 2}{2^4 3^2} \\ &= \frac{45}{144} + \frac{14}{144} = \frac{59}{144}\end{aligned}$$

With algebra

Suppose we want to add the algebraic fractions $\frac{x}{y^4} + \frac{z}{y^3 t^2}$.

We first need to find a common denominator, preferably the least common denominator. The denominators are already factored, and the LCD is $y^4 t^2$.

$$\begin{aligned}\frac{x}{y^4} + \frac{z}{y^3 t^2} &= \frac{x t^2}{y^4 t^2} + \frac{z y}{y^4 t^2} \\ &= \frac{x t^2 + z y}{y^4 t^2}\end{aligned}$$

EXAMPLE 18**With numbers**

Suppose we want to multiply the numeric fractions $\frac{5}{21} \times \frac{33}{45}$.

We first multiply across, then cancel common factors in the numerator and denominator, which can be done by first factoring:

$$\frac{5}{21} \times \frac{33}{45} = \frac{5 \cdot 33}{21 \cdot 45} = \frac{\cancel{3} \cdot \cancel{3} \cdot 11}{\cancel{3} \cdot 7 \cdot \cancel{3} \cdot 9} = \frac{11}{63}$$

With algebra

Suppose we want to multiply the algebraic fractions $\frac{a}{bc} \times \frac{bd}{ab^2}$.

We first multiply across, then cancel common factors in the numerator and denominator, which can be done by first factoring:

$$\frac{a}{bc} \times \frac{bd}{ab^2} = \frac{abd}{bcab^2} = \frac{\cancel{a} b d}{\cancel{b} c \cancel{a} b^2} = \frac{d}{cb^2}$$

ACTIVITY 14

Answer

$$\begin{aligned}1. \frac{42}{70} - \frac{14}{40} &= \frac{2 \cdot 3 \cdot 7}{2 \cdot 5 \cdot 7} \\ &= \frac{\cancel{2} \cdot 7}{\cancel{2} \cdot 2^2 \cdot 5} = \frac{3}{5} - \frac{7}{2^2 \cdot 5} \\ &= \frac{2^2 \cdot 3}{2^2 \cdot 5} - \frac{7}{2^2 \cdot 5} = \frac{12 - 7}{2^2 \cdot 5} \\ &= \frac{5}{2^2 \cdot 5} = \frac{1}{4}\end{aligned}$$

Note that m replaces 2, n replaces 3, p replaces 7, and r replaces 5.

Learning Exercises 12.4

Students do not have answers for Learning Exercises 1, 2(c–g), 3, 4, 5(b–d), 6(b), 7, 8, 10, 12, 14, 15(a), and 16.

**ACTIVITY 14 Your Turn**

- How is finding $\frac{42}{70} - \frac{14}{40}$ like finding $\frac{mnp}{m^2 p} - \frac{mp}{m^3 r}$?
- How is finding $\frac{42}{70} \div \frac{14}{40}$ like finding $\frac{mnp}{m^2 p} \div \frac{mp}{m^3 r}$? ■

The properties of operations together with knowledge of the order of operations allow one to evaluate algebraic expressions and solve algebraic equations. Solving an algebraic equation is the process of using properties of operations and properties of equalities to find the value of the variable(s) that make(s) the equation true.

TAKE-AWAY MESSAGE . . .

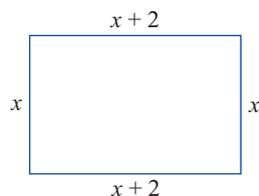
If you understand the underlying reasons for properties and procedures with numbers, then you can generalize those reasons to corresponding situations with polynomial expressions. If you understand the underlying reasons for how we operate with arithmetic fractions, then you can generalize those reasons to corresponding situations with algebraic fractions.

Learning Exercises for Section 12.4

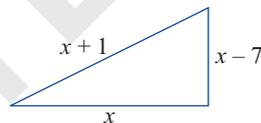
- For each part, write the property or properties being used.
 - Replace $6x + 4x$ with $10x$.
 - Rewrite $(7x + 2) + ax$ as $(7x + ax) + 2$ (two properties).
 - Think of $(15x)(17y)$ as $(15 \cdot 17)(xy)$ (two properties).

- d. Replace $3x + ^{-}7x + ^{-}5x$ with $(3 + ^{-}7 + ^{-}5)x$.
 e. Replace $xy + xz$ with $x(y + z)$.
 f. Replace $(x + 2)(x + 7)$ with $x(x + 7) + 2(x + 7)$.
2. Use the given property or properties to rewrite each expression.
- $(x + 5)x$, commutativity of multiplication
 - $x(4x)$, associativity of multiplication
 - $10n + 10m$, distributive property
 - $y + (3y + 2)$, associativity of addition
 - $(4 - 3)x^3$, identity for multiplication
 - $\frac{4}{3}n^2 + \frac{0}{3}$, identity for addition
 - $\left(\frac{3}{4} \cdot \frac{x}{y}\right) \cdot \frac{y}{x}$, associativity of multiplication, multiplicative inverse property, and identity for multiplication
3. Is 5 a solution of $2x^3 - 3x = 135$? How do you know?
4. Solve each of these equations. Show all your work.
- $3x - 7 = 21.5$
 - $\frac{3}{4}y - 2 = 10$
 - $58 = 4 + (n - 1)3$
 - $\frac{3}{4}(y - 2) = 10$
 - $9x - 12 + 2x = 4x + 16$
 - $20 - 1.95x = 8.3$
 - $4x - 10 = ^{-}7$
 - $^{-}2x - 7 = 11$
 - $\frac{5}{12}t + 3 + \frac{1}{3}t = t - 21$
5. Write an algebraic expression for the perimeter of each shape shown. Simplify the expression, if possible. Assume that the measurements are all in inches.

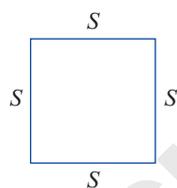
a. rectangle



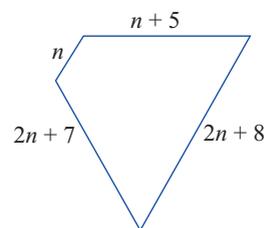
b. triangle



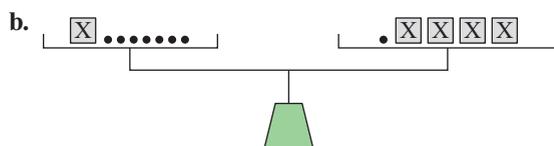
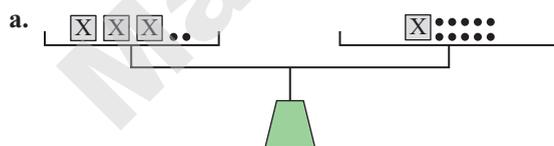
c. square



d. quadrilateral



6. Write equations for the following “balance” displays. A black dot represents 1.



7. How do you solve each of these equations? Compare your algebraic work with actions on a balance representation.

a. $3x + 2 = x + 10$ b. $x + 7 = 1 + 4x$

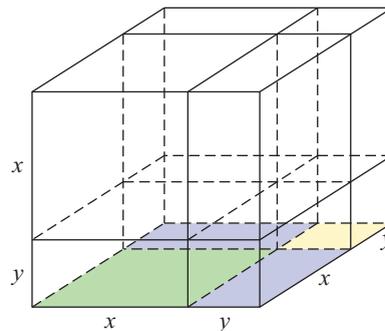
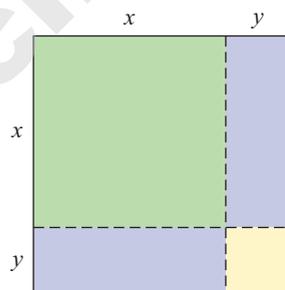
Notes

Learning Exercise 10: Note that this Learning Exercise is different from Learning Exercise 8(a) in that renaming (i.e., “carrying”) is involved.

Learning Exercise 18: This problem requires some memory of how to find area and volume. Learning Exercise 18(a) could be followed up by asking about the square of $(x + y + z)$.

If you are familiar with algebra tiles and have them available, you might wish to introduce them here.

8. Complete the following additions side by side as shown in the examples, and explain how they are alike:
- Add numbers 4026 and 43.
 - Add polynomials $4n^3 + 2n + 6$ and $4n + 3$.
9. Complete the following subtractions side by side, and explain how they are alike:
- $598 - 347$
 - Subtract $3x^2 + 4x + 7$ from $5x^2 + 9x + 8$.
10. Complete the following additions side by side, and explain how they are alike:
- $642 + 188$
 - Add polynomials $6x^2 + 4x + 2$ and $x^2 + 8x + 8$.
11. a. Simplify $\frac{12}{18}$. b. Simplify $\frac{4x^3}{7x^2}$.
12. Complete the following problems, and describe how they are alike:
- $\frac{5}{16} + \frac{7}{16}$
 - $\frac{3x}{y} + \frac{2x+1}{y}$
 - $\frac{5}{x+2} + \frac{x}{x+2}$
13. Complete the following problems, and describe how they are alike:
- $\frac{5}{8} + \frac{3}{4}$
 - $\frac{3x}{(x+2)(x+3)} + \frac{2}{x+2}$
14. Complete the following problems, and describe how they are alike:
- $\frac{7}{9} - \frac{2}{9}$
 - $\frac{2x}{7y} - \frac{4}{7y}$
15. Complete the following problems, and describe how they are alike:
- $\frac{3}{4} - \frac{1}{7}$
 - $\frac{3}{xy} - \frac{4x}{9n}$
16. Complete the following problems, and describe how they are alike:
- $\frac{2}{3} \times \frac{5}{7}$
 - $\frac{2a}{b} \times \frac{c}{3d}$
 - $\frac{4}{x+2} \cdot \frac{2y}{x+3}$
17. Complete the following problems, and describe how they are alike:
- $\frac{5}{6} \div \frac{2}{3}$
 - $\frac{x^2}{2y} \div \frac{xy}{3}$
18. Areas and volumes can give insight into some algebraically equivalent expressions. Using sketches, find or verify equivalent expressions in the following problems:
- $(x + y)^2 = ?$
 - $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$



- $x(x + y) = ?$ (Make your own drawing.)
- $(x + 2)(x + 3) = ?$ (Make your own drawing.)

Supplementary Learning Exercises for Section 12.4

- Give the polynomial suggested by the numeral.
 - 41,025, with $x = 10$
 - 234,125, with $x = 10$

2. Use the numerical calculation to inform the algebraic one. Give the answer to the algebraic one.

$$\begin{array}{r} \text{a.} \quad 513 \\ + 452 \\ \hline \end{array} \quad \begin{array}{r} 5x^2 + x + 3 \\ + 4x^2 + 5x + 2 \\ \hline \end{array} \quad \begin{array}{r} \text{b.} \quad 943 \\ \times 2 \\ \hline \end{array} \quad \begin{array}{r} 9x^2 + 4x + 3 \\ \quad \quad \quad \times 2 \\ \hline \end{array}$$

$$\begin{array}{r} \text{c.} \quad 34 \\ \times 21 \\ \hline \end{array} \quad \begin{array}{r} 3x + 4 \\ \times 2x + 1 \\ \hline \end{array}$$

3. Calculate the sums and products of the polynomials in each part.

$$\begin{array}{ll} \text{a.} & 5x^2 + 3x + 9 \text{ and } 7x^2 + 2x + 10 \\ \text{b.} & \frac{3}{4}x + 2 \text{ and } \frac{1}{3}x + \frac{2}{5} \\ \text{c.} & 5x + (-3) \text{ and } 3x + (-2) \\ \text{d.} & 19x + 25 \text{ and } 22x + 31 \\ \text{e.} & x + 3, 6x + 1, \text{ and } 7x + (-1) \end{array}$$

4. Use the arithmetic calculation to inform the algebraic calculation for each problem, and give answers for the algebraic calculation.

$$\begin{array}{ll} \text{a.} & \frac{2}{5} + \frac{1}{5} \text{ and } \frac{2}{x} + \frac{1}{x} \\ \text{b.} & \frac{2}{5} + \frac{7}{10} \text{ and } \frac{2}{x} + \frac{7}{2x} \\ \text{c.} & \frac{14}{25} - \frac{3}{10} \text{ and } \frac{14}{(y+7)^2} - \frac{3}{2(y+7)} \\ \text{d.} & \frac{1}{25} \cdot \frac{10}{35} \text{ and } \frac{x}{y^2} \cdot \frac{2y}{xy} \\ \text{e.} & \frac{3}{8} \div \frac{9}{16} \text{ and } \frac{z}{n^3} \div \frac{z^2}{n^4} \end{array}$$

12.5 Algebraic Reasoning About Quantities

Many children (and adults) have difficulty solving mathematical story problems. One reason is that many times they are focusing on the numbers in the problem statement, rather than on what the numbers represent. They then try something with the numbers, hoping to get an acceptable answer (and quickly move on to the next problem). Here is misguided advice from a sixth-grade student:⁵

Interviewer: What can we tell students to help them solve story problems?

Ann: If you see a big number and a little number, go for the division. If that doesn't work, then you can try the other ones.

Notice that Ann's focus was on the *numbers*, not on *what the numbers are about*. We distinguish between the numbers and what the numbers are about by using the terms *value* and *quantity*.

A **quantity** is anything (an object, event, or quality thereof) that can be measured or counted. The **value** of a quantity is its measure or the number of items that are counted. A value of a quantity involves a number and a unit of measure or number of units.

For example, your height is a quantity. It can be measured. Suppose the measurement is 64 inches. Note that 64 is a number, and inches is a unit of measure: 64 inches is the value of the quantity, your height. The number of people in a room is another example of a quantity. Suppose the count is 32 people. Note that 32 is a number and the unit counted is people, so 32 (people) is the value associated with the quantity, number of people in the room.

A quantity is not the same as a number. In fact, one can think of a quantity without knowing its value. For example, the amount of snow during a month is a quantity, regardless of whether or not someone measured the actual number of inches of snowfall. One can speak of the amount of snowfall without knowing how many inches actually fell. Similarly, one can speak of a person's annual income, a school's enrollment figure, the speed of a race car, or the amount of time it takes to get to school (all are quantities) without knowing the actual values.

It is important that students understand the difference between the value of a quantity and the quantity itself.

Notes

DISCUSSION 6*Answers*

1. a. height
- b. rate of flow
- c. dollar amount of damage, number of casualties, etc.
2. a. feet, meters, etc.
- b. any unit of volume per any unit of time
- c. number of dollars of damage costs, number of lives

Chapters 1, 8, and 9 also emphasize quantitative reasoning in contexts.

SEE INSTRUCTOR NOTE 12.5A for an example that could be used to demonstrate solving a problem quantitatively, perhaps before Example 19.

EXAMPLE 19

Admittedly, most students do not wish to do this much “work.” But point out that if they are stuck on a problem, focusing on the quantities, their values, and the relationships among these quantities is helpful. In Step 3 of the solution, you may have to talk about the diagram. It was made, of course, gradually and by repeatedly referring to the problem statement.

**DISCUSSION 6** Identifying Quantities and Measures

1. Identify the quantity or quantities addressed in each of the following questions.
 - a. How tall is Mount Everest?
 - b. How fast does gasoline come out of a pump?
 - c. How much damage did the storm cause?
2. Identify an appropriate unit of measure that can be used to determine the value of the quantities involved in answering the questions in Problem 1. ■

The difficulty with story problems becomes greater with algebra story problems, because some relevant values may not even be apparent. To write an equation for the problem, it becomes essential to focus on the quantities involved and how the quantities are related.

In this section, you will analyze problem situations in terms of the quantities and their relationships. Such analyses are essential to being skillful at solving mathematical problems.

For the purposes of this course, to **understand a problem situation** means to recognize the quantities embedded in the situation and understand how the quantities are related to one another.

Understanding a problem situation “drives” the solution to the problem. Without such understanding, the only recourse a person has is to guess at the calculations that need to be performed or to proceed as sixth-grader Ann does. It is important that you work through the following problems with care and attention. Analyzing problem situations quantitatively is a powerful tool that will help you become a better problem solver.

EXAMPLE 19

Try this problem before you read the solution. Then compare your solution to the one shown here.

- Three boys spent the afternoon playing a video game and then compared their best scores. Al says, “My best today is 900 points more than yours, Bob.” Carlo says, “I was having a good day. My 3600 points is $\frac{2}{3}$ as much as your two scores combined, Al and Bob.” How many points did each boy get in his best game that afternoon? ◀

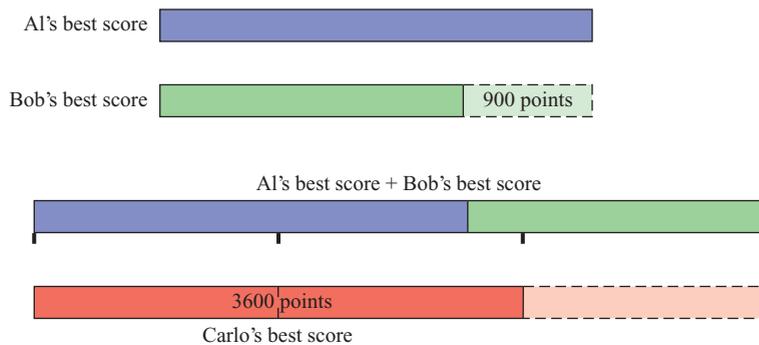
Possible Solution

1. Name the quantities involved in the problem.
 - a. Al’s best score
 - b. Bob’s best score
 - c. Carlo’s best score
 - d. the difference between Al’s and Bob’s best scores
 - e. the total of Al’s and Bob’s best scores
 - f. the comparison of Carlo’s best score to Al and Bob’s total
2. Identify the values of the quantities.

a. Al’s best score	unknown
b. Bob’s best score	unknown
c. Carlo’s best score	3600
d. the difference between Al’s and Bob’s best scores	900
e. the total of Al’s and Bob’s best scores	unknown
f. the comparison of Carlo’s best score to Al and Bob’s total	$\frac{2}{3}$
3. Look for relationships. This step often involves rereading the problem and making a diagram for the problem. The second sentence relates quantities (a), (b), and (d). Al’s best score is 900 more than Bob’s. The third sentence relates quantities (c), (e), and (f):

$$3600 = \frac{2}{3} (\text{Al's best score} + \text{Bob's best score})$$

A possible diagram follows:



4. If you use algebra to solve the problem, introduce a variable and try to write an equation. (As you may know, some problems require more than one variable, but these problems are not usually encountered until well into a study of algebra.) It is not clear in this problem which score should be designated as the unknown. Let us try x as the number of points in AI's best score. Then Bob's must be $x - 900$. Carlo's information then leads to the equation $3600 = \frac{2}{3}(x + x - 900)$, which leads to $5400 = 2x - 900$. Solving this equation, we arrive at $6300 = 2x$, which in turn leads to $x = 3150$. So AI's best score was 3150 points, Bob's was 2250 points (from $3150 - 900$), and Carlo's was 3600 points (known). How did your solution compare with the elaborated one? ●

Of course, usually we do not write so much detail because some mental work is involved. But when you are stuck, try writing down more and referring to your diagram for help in seeing relationships. In Example 19 you may even have noticed that you could solve the problem without algebra.

THINK ABOUT ...

Suppose that in Example 19 you had let x be the number of points in Bob's best game. What equation would you have arrived at in that case? Would you have gotten the same answers for the given problem?

Often, students begin a problem such as the one in Example 19 by asking themselves, "What operations do I need to perform, with which numbers, and in what order?" Instead of these questions, they would be much better off asking general questions, such as the ones that follow:

Quantitative Analysis Questions

- Can I imagine the situation, as though I am acting it out?
- What quantities are involved? What do I know about their nature? Do some change over time? Do some stay constant?
- What is the quantity of interest? That is, what quantity and its value am I asked to find or describe?
- How could making a diagram (or a sequence of diagrams) help reveal relationships among the quantity of interest and other quantities?
- What quantities and values do I know that could help me find the value of the quantity of interest?
- What quantities and values can I derive to help me find the value of the quantity of interest?

Notes

Placing a variable in the box for either AI's or Bob's best score can help students see equations for the problem.

Notes

Quantitative Analysis Questions: This list of questions is *extremely* important. Whenever you find your students struggling to understand a problem situation, come back to this list, and make sure they are asking themselves these types of questions and really trying to make sense of the situation.

EXAMPLE 20

You may recognize that these speeds are fictional.

MathClips 

-  Snail Love Part 1
-  Snail Love Part 2

The point is that you need to be specific when doing a quantitative analysis of a given situation. How specific? It is difficult to say in general, because it will depend on the situation. Use your common sense in analyzing the context. When you first read a problem, try not to think about only the numbers and the operations of addition, subtraction, multiplication, and division. Instead, start by posing questions such as the quantitative analysis questions and trying to answer them. Once you've done that, you will have a better understanding of the problem, and thus you will be well on your way to solving it. Pólya described four principles for solving problems:

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

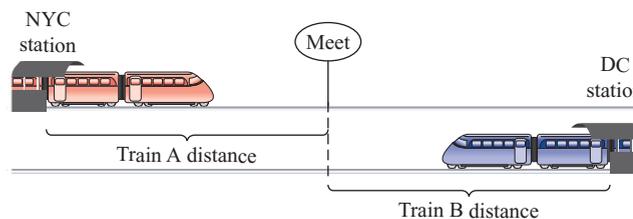
Understanding a problem is the first step—as well as the most difficult aspect—in solving it. As in Example 19, if you draw a diagram and label it with variables, you can see how equations can model the situation and make it easier to solve.

EXAMPLE 20

This problem is an example of a word problem typically found in an algebra textbook. Often, word problems can be solved quite easily when students list all the quantities to be used and draw a diagram. But here we use a quantitative analysis and a diagram to guide two nonalgebraic solutions and an algebraic solution.

- Amtrak trains provide efficient, nonstop transportation between New York City and Washington, D.C. Suppose Train A leaves New York City headed toward Washington at the same time that Train B leaves Washington headed for New York City, traveling on parallel tracks. Train A travels at a constant speed of 65 miles per hour. Train B travels at a constant speed of 75 miles per hour. The two Amtrak stations are 210 miles apart. How long after they leave their respective stations do the trains meet? ◀

Quantities	Values
Speed of Train A	65 mph
Speed of Train B	75 mph
Distance traveled by Train A to meeting point	? miles
Distance traveled by Train B to meeting point	? miles
Total distance traveled by each train	210 miles
Time traveled by both trains, departure to meeting point	? hours



Here is a traditional algebraic solution that follows from a quantitative analysis and a drawing. In this solution, note that the time traveled by both trains is the same, which is t hours, and that students use the $d = rt$ formula, with Distance_A and Distance_B

representing the distances (in miles) traveled by Trains A and B, respectively, when the two trains meet.

$$\begin{aligned} \text{Distance}_A + \text{Distance}_B &= 210 \text{ miles} \\ 65t + 75t &= 210, \text{ so } 140t = 210 \text{ and } t = 210 \div 140 = 1.5 \text{ hours} \end{aligned}$$

You may have noticed that algebra is not essential for this particular problem and would not be needed in elementary school. Here is one nonalgebraic solution: In 1 hour, the trains together will go $65 + 75 = 140$ miles. Since they need to go 210 miles, they will meet after $210 \div 140 = 1.5$ hours. A second nonalgebraic solution might look at how far each train goes in 1 minute: $\frac{65}{60}$ miles and $\frac{75}{60}$ miles, or $\frac{65}{60} + \frac{75}{60} = \frac{140}{60}$ miles together. Again, since they need to go 210 miles, the calculation $210 \div \frac{140}{60}$ will give the number of minutes needed. ●

Here is a final example, for which quantitative reasoning is awkward for some solvers.

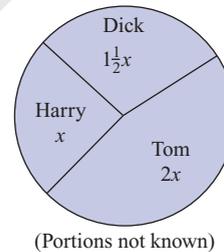
EXAMPLE 21

► Three brothers, Tom (age 16), Dick (age 14), and Harry (age 9), were home alone and hungry. They decided to buy a whole apple pie and eat it. They chipped in their money—Tom, \$4; Dick, \$2; and Harry, \$3—and Tom bought a large pie. They succeeded in eating it all. Tom ate twice as much as Harry, and Dick ate $1\frac{1}{2}$ times as much as Harry. What part of the pie did each brother eat? ◀

Solution

Some of the quantities—the brothers' ages (except to justify the unequal shares), the amounts of money chipped in, the total cost of the pie—seem irrelevant to the question. The relevant quantities are the brothers' shares of the pie. The shares are based on Harry's share (and not the money chipped in), so let x be the fraction of the pie eaten by Harry. Then Tom's fraction is $2x$, and Dick's is $1\frac{1}{2}x$.

The sketch shows it: The three brothers ate the whole pie, so $x + 2x + 1\frac{1}{2}x = 1$, and hence $4\frac{1}{2}x = 1$, or $x = \frac{2}{9}$. Harry ate $\frac{2}{9}$ of the pie, Tom ate $\frac{4}{9}$ of the pie, and Dick ate $\frac{3}{9} = \frac{1}{3}$ of the pie. ●



There are other heuristics or problem-solving strategies that are useful. Working backward is an example of a problem-solving heuristic. So, too, is finding a pattern or solving a simpler related problem. We used the strategy of solving a simpler problem and looking for a pattern in earlier sections, for example, Activity 8 and Activity 11. Other examples of problem-solving heuristics are doing an educated guess-and-check, using a mathematical model, and considering special cases. Noticing a pattern is often used along with guess-and-check to aid in problem solving.

Try solving Activity 15 before looking any further. Share your different strategies.

ACTIVITY 15 What to Sell?

► At the Comic-Character conference, you can sell from a booth. You have \$1000 credit to spend on figurines (and you want to use it all) to sell in your booth. You have room for 144 figurines. Arachnid Man figurines cost \$5 each, and Ferrous Woman figurines cost \$10 each. If you buy 144 figurines and spend \$1000, how many of each figurine did you have for sale at your booth when the conference opened? ◀

Notes

EXAMPLE 21

SEE INSTRUCTOR NOTE 12.5B for a strictly quantitative solution.

Notes

There are many ways to solve this problem. Perhaps you used guess-and-check and then noticed a pattern. Here is one sixth-grader's successful solution: He began with 72 of each figurine and noticed that one is twice the cost of the other, so a change of two Arachnid Men for two Ferrous Women lowers the cost by \$10. He knew he needed 7 more pairs to exchange in order to lower the cost by \$70.

Number of Arachnid Man figurines	Number of Ferrous Woman figurines	Total cost
72	72	$72 \cdot \$5 + 72 \cdot \$10 = \$1080$
74	70	$74 \cdot \$5 + 70 \cdot \$10 = \$1070$
...
88	56	$88 \cdot \$5 + 56 \cdot \$10 = \$1000$

There's also an algebraic solution, of course. Just as a diagram in the earlier problems suggested the equation, your informal reasoning and the table can suggest how to set up the algebraic equations. The sum of the number of Arachnid Man figurines (a) and the number of Ferrous Woman figurines (f) is 144: $a + f = 144$. The total cost is \$1000: $5a + 10f = 1000$. Perhaps you are unsure how to solve equations with two variables. We will be learning more about this in later chapters.

Undertake a quantitative analysis to understand the problem and devise a plan! Use your observation of important mathematical relationships to build mathematical models (e.g., diagrams, tables, or equations) to devise a solution. Check the reasonableness of your answer in context, and adjust your plan if necessary.

In the Learning Exercises for this section, you will find many problems that can be solved either arithmetically or algebraically. A diagram is usually useful in either case. But for many problems, algebra is required, or at least makes the problem easier to solve.

TAKE-AWAY MESSAGE ... Using the quantitative analysis questions, you should be able to determine the quantities and their relationships within a given problem situation and use this information, together with a diagram as needed, to solve problems. The following steps are often useful:

1. List the quantities that are essential to the problem.
2. List known values for these quantities.
3. Determine the relationships involved, which is frequently done with the help of a diagram.
4. Use the knowledge of these relationships to write an equation to solve the problem.

This approach may not seem easy at first, but it becomes a powerful tool for understanding problem situations. When used with elementary school students, this focus on quantities prepares them for algebra story problems.

Learning Exercises for Section 12.5

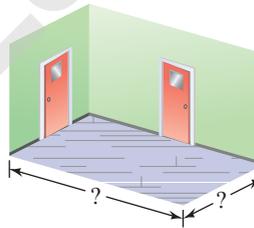
Solve each problem. Diagrams are recommended, and the use of algebra is encouraged. If you use algebra, make clear what quantity your variable represents.

1. Dan, Lincoln, and Miguel are pooling their cash to see whether they have enough to buy a game. They have a total of \$46.80. They notice that Lincoln has twice as much as Dan and that Miguel has three times as much as Dan. How much does each boy have?
2. Your school's student council held a dance after a basketball game. Tickets were \$2 each for students and \$3 each for nonstudents. You know that there were 6 nonstudents at the dance and that, overall, the student council made \$148. How many students attended the dance?

Learning Exercises 12.5

Students do not have answers for Learning Exercises 2, 3, 4, 7, 8, 10, 12, 13, 14(b), and 15.

3. The Jones family takes a car trip every December. Last year, Lee wanted to record the miles driven each day for a report. The family drove for 5 days: 316 miles during the second driving day, 280 miles during the third, and 156 during the fourth. Lee forgot to record the mileage during the first and fifth days. Lee's dad knew that Lee had been studying algebra, so he said, "I kept track, too. We drove a total of 1112 miles. And we drove twice as many miles on the fifth day as we did on the first day because we got a late start the first day. So now you can figure out the missing mileages." What might Lee's algebraic solution look like?
4. Puzzle problems about coins often appear in algebra books to give students practice in representing situations algebraically.
- Beth notices that she has several coins in her dresser drawer, worth \$1.41 in all. She has 3 more nickels than dimes, 16 pennies, and 2 quarters. How many nickels and dimes are there? (*Hint*: Notice that you must distinguish between two quantities: the *number* of each type of coin, and the *monetary value* of those coins. If there are n dimes, their value is $10n$ cents.)
 - Serena notices that she has a lot of change in her purse. She has twice as many dimes as quarters, 6 more nickels than dimes, and 18 pennies. In all, she has \$2.13. How many of each type of coin does she have?
 - Shawna has been collecting her change for some time and has 101 coins in all. She has 6 half-dollars, half as many quarters as dimes, 85¢ in nickels, and 3 times as many pennies as nickels. How much is her collection worth? (*Hint*: First find out how many coins of each kind she has.)
 - Make up your own coin problem.
5. You are measuring a rectangular room for painting. You notice that the room is exactly twice as long as it is wide and that the total distance around the room is 87 ft. What are the dimensions (the length and width) of the room?
6. A town has been using a recycling plan for three years. The second year of the plan saw an increase in recycled tonnage of 25% over the first year, and the third year saw an increase of 20% over the second year. The mayor proudly claims, "During those three years, we have saved 41,250 tons of trash from being put into our landfill." How many tons were recycled each of the three years? (*Hint*: For the third year, you will probably want an expression for 120% of the expression for the second year.)
7. One type of problem often found in algebra textbooks is called a "work problem." Here is one:
- If Thuy can paint her family room in 6 hours and her husband can paint it in 8 hours, give a good estimate of how long it should take to paint the room if they work together (under ideal circumstances). ◀



- Give both a good estimate based on a diagram and an exact answer using algebra.
8. Callie left home at 7:30 AM to bike 8 miles to the school where she teaches. Luckily, she could ride on a bike trail where she averaged 12 miles per hour. But at 7:40 AM her husband Jeff noticed that she had forgotten to take her graded exam papers, which he knew she wanted to return that day. So he hopped on his bike to catch up with her. If he averaged 16 miles per hour, did he catch her before she arrived at school? If so, what time did he catch up with her? (*Hint*: How far could each ride in 1 minute?)
9. Officials from two cities, Allswell and Bestburg, are talking about their anticipated tax revenue for the coming year. The Allswell official says, "Our tax revenue will be up 20%, or \$15,000,000, over this year's." The Bestburg official says, "Our revenue will be up by 25%, and even then we will have \$10,000,000 less than you do." What is this year's tax revenue for Bestburg?

Notes

Most school algebra problems are called puzzle problems because they can be answered without algebra and often do not even represent situations of recognizable importance. In Learning Exercise 1, for example, the boys already know how much money each has. So you may wish to acknowledge this state of affairs, but at the same time point out that such problems offer a good opportunity to represent situations algebraically. And some students find them to be fun!

Learning Exercise 8: This problem is most easily solved via a drawing. Algebra is not particularly helpful.

Notes

10. Dollie decides to drink just water after finding out that she was consuming too much caffeine—more than 330 milligrams a day. On a typical day, she drank a cup of coffee (120 milligrams of caffeine) and two cans of diet cola in the morning (each containing 42 milligrams of caffeine) and more cans of the same diet cola after noon. How many cans of diet cola did she usually drink after noon?
11. Your car repairs cost \$548.95, for parts and labor. The parts cost \$338.95 and labor was \$35 an hour. How many hours of labor did the repairs take?
12. You joined a neighborhood health club. The one-time initiation fee was \$50, and the monthly fee is \$30. You also bought one package of personal fitness consultations at \$100 for four 25-minute sessions. You wrote a check for \$240. How many months did you pay for?
13. These problems are from a 1924 book of arithmetic story problems.⁶ (Capitalization was used more frequently in 1924, as this original text shows.) They are much easier to solve using algebra ideas, like arithmetic sequences.
- a. “Ignatius Trott had, at the age of 7 years, a Conscience able to support a weight of 22 lb., 4 oz. The weight which it could carry increased every year thereafter 16 lb., 7 oz. What weight can his Conscience support at his present age, $47\frac{1}{2}$ years?” (If you have forgotten, 1 pound or 1 lb = 16 ounces or 16 oz. *Advice:* Work with fractions.)
- b. “A Boy and his Sister, ages 7 and 8, are eating Watermelons. Working jointly, it takes them 5 minutes to consume the first Melon, 10 minutes the second, 15 minutes the third, etc., etc. What will be the number of the Melon which will require 1 hour and 35 minutes for its consumption?”
14. Gabby and Helen try to stay in shape by working out on an exercise bicycle. Each woman sets a target for a Monday through Friday week.
- a. One week Gabby biked the following times:
- Tuesday:* 5 minutes more than on Monday
Wednesday: twice as long as on Tuesday
Thursday: twice as long as on Monday
Friday: same time as on Monday
- Gabby met her target of 2 hours for that week. How many minutes did Gabby bike each day?
- b. Helen sets her goal in terms of *miles* for the week:
- Monday: 4 miles
 Wednesday: 1 fewer mile than on Tuesday
 Thursday: 3 times as far as on Tuesday
 Friday: twice as far as on Wednesday
- Helen met her goal of 50 miles. How many miles did she bike each day?
15. “Our town’s budget of \$19.8 million is up 20% over last year’s budget, and last year’s budget was 10% more than the previous year’s! We taxpayers are angry.” What were the town’s dollar budgets for last year and the year before that?

Supplementary Learning Exercises for Section 12.5

The parts in each numbered problem follow roughly the same theme but may not give completely similar equations.

1. a. Paula and Ruell receive the same weekly allowance. One Saturday after receiving their allowances, Paula spent \$5 and Ruell spent \$2. Paula then had only half as much left as Ruell had left. How much was each allowance?

- b. Ray and Amir each bought a used car for the same price. Each needed some repairs. The repairs on Ray's car cost an additional \$800 and on Amir's, an extra \$200. Amir noted, "Ray, my total cost was really only $\frac{5}{6}$ of your total cost." How much did each pay for his used car?
- c. Silvia and Mai bought complete outfits for the same price. Silvia returns the fancy \$20 belt, and Mai returns her \$45 shoes. Mai said, "Silvia, I actually spent only $\frac{3}{4}$ as much as you did." How much did each original outfit cost?
2. a. A (four-member) relay team ran the mile relay. The second runner was 2 seconds slower than the first runner, and the third runner took 3 seconds longer than the second runner. The fourth runner ran 4 seconds faster than the second runner. Their total time was 3:59 (3 minutes, 59 seconds). What was the time for each of the four runners?
- b. A (four-member) relay team swam the 400-meter relay in a time of 6:53 (6 minutes, 53 seconds). The second swimmer took 2.5 seconds longer than the first swimmer, and coincidentally, the third and fourth swimmers took the same total amount of time as the first two did. What was the time for each of the first two swimmers?
- c. In one grand slalom ski race, Bjorn won with a total time of 1:40.68 (1 minute, 40.68 seconds) for two runs, by skiing 1.06 seconds faster on his second run. What was his time for each run?
3. a. Three friends went shopping. Danetta spent \$25 more than Elaine spent, but Jan spent \$35 less than Danetta spent. If they spent \$165 in all, how much did each of the friends spend?
- b. The three friends went shopping again. This time Danetta spent \$12 less than Jan spent, but Elaine spent twice as much as Danetta spent. They spent \$86 in all. How much did each friend spend this time?
- c. Make up a shopping problem, and solve it algebraically (even though you perhaps made up the problem by thinking at the start of amounts spent by each person).
4. a. You went shopping at three stores and spent \$167.80 in all. You spent three times as much at the first store as you spent in all at the other two stores. You spent \$8.45 more at the second store than you spent at the third store. How much did you spend at each store?
- b. Sarita shopped at four stores and spent a total of \$49.65. She spent \$15.95 more at the third store than she spent at the first store, and \$8.75 less at the fourth store than at the first store. Sarita did not buy anything at the second store. How much did she spend at each store?
- c. Make up a shopping problem, and solve it algebraically (even though you perhaps made up the problem by thinking at the start of the amounts spent at the different stores).
5. a. Alima went for a $2\frac{1}{2}$ -mile walk with her sister Noor, Noor's husband, and their baby. The baby was heavy, so they took turns carrying him. Alima carried the baby only one-third as far as her sister, and Noor's husband carried the baby twice as far as Noor did. How far did each person carry the baby?
- b. Dien, Gia, and Minh took a vacation together, sharing the driving. One day they drove 8 hours and 35 minutes. Minh drove twice as long as Gia did, and 10 minutes longer than Dien did. How long did each person drive that day?
- c. On another day, Dien, Gia, and Minh drove 475 miles. Gia drove twice as far as Dien did, and Minh drove 115 miles. How far did each person drive that day?
6. a. A store manager notices that sales for last year were 25% more than the previous year. She knows that sales for last year were \$840,000. What were the sales in dollars the previous year?
- b. A clothing shop had sales of \$180,000 over a two-year period. Sales during the second year were 40% higher than sales during the first year. What were the sales figures for each of the two years?

Notes

- c. Tabatha has owned a small store for three years. The store has grossed a total of \$1,200,000 over those years. The gross sales for the second year were 20% more than those for the first year, and sales for the third year were 80% more than those for the first year. What were the gross sales for those three years?
7. a. Susan, Rachel, and Steve made 6 dozen cookies. Over a period of days, they ate them all. Susan ate 25% more cookies than Rachel ate, and Steve ate 2 more cookies than Susan ate. How many cookies did each of the three eat?
- b. Rayann made 2 dozen deviled eggs for a family gathering, but only she and her two cousins, Nell and Bell, ate any. Together, they ate $\frac{2}{3}$ of the eggs. Rayann ate one-third as many as Nell ate. Bell ate one more than Rayann ate. How many deviled eggs did each eat?
- c. Anh, Rama, and Amy together ate all of a licorice whip that was 1 yard long. Anh ate part of it in 2 minutes at the rate of 4 inches a minute, Rama ate for $1\frac{1}{3}$ minutes at a rate of 1 foot per minute, and Amy finished the whip in 3 minutes. What was Amy's rate of eating, in inches per minute? (Note: 1 yard = 3 feet; 1 foot = 12 inches)
- d. Steve and Susan went on a hike on a hot day, taking with them 2 quarts of water. During the hike, Susan drank a half pint more than Steve did. They drank all the water. How much water did each of them drink? (Note: 1 quart = 2 pints)
8. Rushmore Park has a walking/running track. Carita and Jorge begin walking at the same place at the same time. Carita consistently walks 6 miles per hour and Jorge walks 4 miles per hour. After a while, Carita shouts to Jorge, "Jorge, I'm a quarter mile ahead of you." How many minutes had they been walking when Carita said that?

12.6 Issues for Learning: The National Assessment of Educational Progress and Achievement in Algebra

The National Assessment of Educational Progress (NAEP, usually pronounced "nape") has been recording achievement at the fourth-, eighth-, and twelfth-grade levels for several decades. Measures of mathematics achievement over the years provide a snapshot of how well the students in our country are performing. Only a few of the test items are released because, once released, items cannot be used again. The released items, and performance on these items, can be seen at <http://nces.ed.gov/nationsreportcard/>. All the test items discussed in this section were used in the 2011 testing.

On the fourth-grade level, algebra questions assess students' understanding of many of the topics covered in this chapter. Students must recognize and extend patterns, as well as use symbols to represent unknown quantities. The following test items were used in the fourth-grade algebra assessment.

Test Item 1. Sam folds a piece of paper in half once.

There are two sections.

Sam folds the piece of paper in half again.

There are four sections.

Sam folds the piece of paper in half again.

There are eight sections.



Sam folds the piece of paper in half two more times. Which list shows the number of sections there are each time Sam folds the paper?

- (A) 2, 4, 8, 10, 12 (B) 2, 4, 8, 12, 24 (C) 2, 4, 8, 16, 24 (D) 2, 4, 8, 16, 32

Recognizing and extending this pattern were very difficult for fourth-grade students. Only 23% were able to choose D as the correct answer, though this is a straightforward doubling pattern.

Test Item 2. Every 30 minutes, Dr. Kim recorded the number of bacteria in a test tube.

Time	1:00 PM	1:30 PM	2:00 PM	2:30 PM
Number of bacteria	600	1190	2390	4800

Which explanation best describes what happened to the number of bacteria every 30 minutes?

- (A) The number of bacteria increased by 500.
- (B) The number of bacteria increased by 1000.
- (C) The number of bacteria doubled.
- (D) The number of bacteria tripled.

The doubling pattern in Test Item 2 was still difficult to recognize in a table, but 34% of fourth-graders correctly chose answer C.

Test Item 3. Each of the 18 students in Mr. Hall's class has p pencils. Which expression represents the total number of pencils that Mr. Hall's class has?

- (A) $18 + p$
- (B) $18 - p$
- (C) $18 \times p$
- (D) $18 \div p$

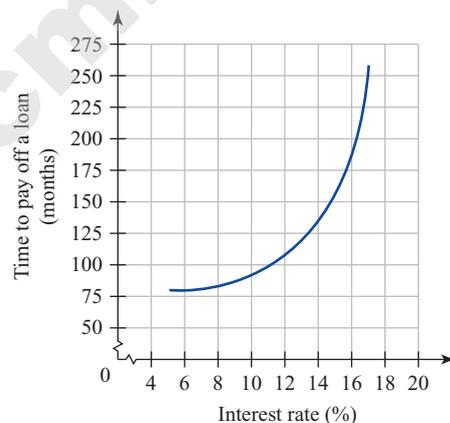
Students in fourth grade are asked to identify an expression that models a context. This was also a difficult question; 35% of the students answered it correctly. Knowing that C is the correct answer demonstrates an understanding of the operation of multiplication as repeated addition. A student must also understand how the symbol p can stand for a variety of values.

Students in eighth grade were generally more capable of identifying which equation was appropriate for a given context. More than half (53%) of the eighth-graders correctly identified C as the answer to Test Item 3.

Test items pertaining to other areas of algebra are also available. The eighth-grade test covers content usually included in fifth grade through seventh grade. Not all algebra test items involve variables. Interpreting graphs is a very important topic. We will explore graphs in subsequent chapters. The following NAEP item is one such problem. Approximately 70% of the eighth-graders could give the correct answer for this item.

Test Item 4. According to the accompanying graph, between which of the following pairs of interest rates will the *increase* in the number of months to pay off a loan be greatest?

- (A) 7% and 9%
- (B) 9% and 11%
- (C) 11% and 13%
- (D) 13% and 15%
- (E) 15% and 17%



There are, of course, many other items on the NAEP test. NAEP results have become more important in the last decade because they provide not only national results but also individual state results. Hence, they serve as a way of comparing progress at the national and state levels from one test administration to the next. Thus, it is good to be acquainted with the types of items appearing on the NAEP test and perhaps to try some of them in your own classrooms.

12.7 Check Yourself

After studying this chapter, you should be able to do problems like the ones assigned and to meet the following objectives:

1. Name and use the properties involved in given numerical or algebraic work.
2. Evaluate algebraic expressions, and solve simple equations.
3. Give parallel numerical and algebraic calculations, and point out how they are alike.
4. Calculate the sum and product of two polynomials.
5. Illustrate, identify, and work with arithmetic sequences and geometric sequences.
6. Find a specific number, or an expression for the n th number (or a function rule), in a given pattern or table. Generate your own data for some situations.
7. Explain what a function is and why functions are important in mathematics.
8. Explain why a function rule based only on examples might not be 100% reliable.
9. For selected situations, find a general function rule and give a justification that it is 100% reliable.
10. Solve a story problem using algebra (and a diagram).

References for Chapter 12

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